Logical and universal algebraic geometries, general algebraic groups, and non-standard models

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Outline

In this talk I will focus on three things:

- Universal and logical algebraic geometries, their advantages and limitations.
- Interpretations as the main tool to study logical algebraic geometry and a general notion of an algebraic group (ring, etc).
- The first-order classification problem and non-standard groups (rings, etc).

The universal algebraic geometry

The universal algebraic geometry is an analog of the classical algebraic geometry (over $\mathbb{C}$) developed for arbitrary algebraic structures.

For example, in the case of groups the principal objects in this geometry are systems of equations, their solution sets (algebraic sets) and the corresponding coordinate groups.

The fundamental research questions concern with algebraic characterization of the coordinate groups, equational Noetherian property, Krull dimension, Nullstellensatz, etc.

The basics of the universal algebraic geometry was introduced in papers by B.Plotkin, G. Baumslag, A. Myasnikov, V. Remeslennikov and O. Kharlampovich.

Algebraic geometry over groups was successfully developed for several large classes of groups:

- Free and hyperbolic groups
- Partially commutative groups
- Free solvable and rigid groups

In all these cases two things played an important part: the equational Noetherianess and detailed study of the algebraic structure of the coordinate groups.

The logical (algebraic) geometry was introduced rather recently by B. Plotkin.

In this geometry the principal objects are sets of arbitrary first-order formulas, their solutions sets, and the coordinate algebras.

Since equations are first-order formulas (though very simple ones) the logical algebraic geometry is a generalization of the universal one.

The first comparison

By design universal algebraic geometry concerns mostly with algebra problems and its methods are mostly algebraic.

The logical algebraic geometry relates to model theory and model-theoretic algebra, its methods are mostly in the realm of algebraic logic and model theory.

The first glance at the logical geometry

The logical geometry is a new subject in its early stages, so it is too early to discuss how powerful or useful it is in solving problems.

However, it seems clear already that this geometry gives an unusual view-point and interesting approach to some well-known problems.

In what follows I will quickly discuss some interesting ideas that come from the paradigm of the logical geometry and some crucial differences with the universal geometry.

As was mentioned above there are two fundamental properties that every good universal geometry enjoys:

- Equational Noetherianess
- The coordinate groups have reasonably nice algebraic properties

Let's see what happens in this respect in logical algebraic geometry.

Recall the following

Definition

A group G (ring, etc) is equationally Noetherian if every system of group equations in finitely many variables is equivalent in G to some finite part of itself.

One can define an analog if this notion in the logical algebraic geometry.

Recall, that an n-type in G is a set of first-order formulas in variables $x_1, \ldots, x_n$ that is consistent with the theory $Th(G)$.

A type p is complete if for every formula $\phi(x_1, \ldots, x_n)$ in group language either $\phi \in p$ or $\neq \phi \in p$.

Complete types define irreducible algebraic sets.

A type p is isolated (or principle) if there is a formula ϕ in p such that all other formulas in p are consequences in $Th(G)$ of the formula ϕ.

Logical equational Noetherianess

Definition [Plotkin]

A structure G is logically Noetherian if every type realized in G is isolated.

However, this notion is rather restrictive. It seems it suffices to consider only irreducible algebraic sets.

Definition [Almost logical Noetherianess]

A structure G is almost logically Noetherian if every complete type realized in G is isolated.

Logical equational Noetherianess

It turns out that almost logically Noetherian structures are well-known in model theory.

Atomic structures

A structure G is atomic if every complete type realized in G is isolated.

Definition [Prime models]

A structure G is prime if it is an elementary substructure of every structure that is first-order equivalent to G.

Properties of atomic models:

- An infinite structure is prime iff it is countable and atomic.
- Every prime structure G is homogeneous, i.e., two tuples in G realize the same type iff they are conjugate by an automorphism of G.

Examples of prime structures

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are prime, so almost logically Noetherian.

Note, that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are not logically Noetherian.

A known result from model theory

Every ω-stable complete theory in countable language has a prime model.

Algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ is prime, so almost logically Noetherian.

Algebraic groups

Algebraic groups G over algebraically closed fields are ω-stable, i.e., their theories $Th(G)$ are ω-stable.

It was noticed that a more general fact is true:

Known results

- Structures interpretable in an algebraically closed field are ω-stable.
- More generally, structures interpretable in ω-stable structures are ω-stable themselves.

The classical algebraic groups are defined by "schemes" from the universal algebraic geometry, i.e., when algebraic sets are defined by systems of equations.

What is a direct analog of an "algebraic group" in the logical algebraic geometry?

Definition

A structure $\mathbb{A}$ is algebraic over a structure $\mathbb{B}$ if $\mathbb{A}$ is interpretable in $\mathbb{B}$.

Definition

An algebraic structure $\mathbb{A} = \langle A; f, \ldots, P, \ldots, c, \ldots \rangle$ is absolutely interpretable (or 0-interpretable) in a structure $\mathbb{B}$ if there is a subset $A^* \subseteq B^n$ definable in $\mathbb{B}$, an equivalence relation $\sim$ on A^* definable in $\mathbb{B}$, operations $f^*, \ldots,$ predicates $P^*, \ldots,$ and constants $c^*, \ldots,$ on the quotient set $A^*/\sim$ all interpretable in $\mathbb{B}$ such that the structure $\mathbb{A}^* = \langle A^*/\sim; f^*, \ldots, P^*, \ldots, c^*, \ldots, \rangle$ is isomorphic to $\mathbb{A}$.

Bi-interpretability

Let $\mathbb{A}$ and $\mathbb{B}$ be two structures, possibly of different signatures.

Suppose that $\mathbb{B}$ is interpeted in $\mathbb{A}$ as $\mathbb{B}^*$ and $\mathbb{A}$ is interpreted in $\mathbb{B}$ as $\mathbb{A}^*$ such that

- for interpretations

$$\mathbb{A} \succ \mathbb{B}^* \simeq \mathbb{B} \succ \mathbb{A}^*$$

  there is an isomorphism $\lambda : \mathbb{A} \to \mathbb{A}^*$ definable in $\mathbb{A}$.

- for interpretations

$$\mathbb{B} \succ \mathbb{A}^* \simeq \mathbb{A} \succ \mathbb{B}^*$$

  there is an isomorphism $\mu : \mathbb{B} \to \mathbb{B}^*$ definable in $\mathbb{B}$

In this case we say $\mathbb{A}$ and $\mathbb{B}$ are bi-interpretable in each other.

Bi-interpretability and prime models

Theorem [KMS]

An infinite structure bi-interpretable with a prime structure is prime.

Corollary

An infinite structure bi-interpretable with $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Q}$, is prime.

This gives a huge amount of prime structures, in particular prime groups and rings.

Looking again at the groups we discussed above.

The following groups are prime:

- $BS(1, m)$, $m > 1$ and $\mathbb{Z}_n wr \mathbb{Z}$, $n > 1$.
- Free metabelian of finite rank > 1.
- Thompson group F.
- $SL(n, \mathbb{Z})$ and $SL(n, \mathbb{Q})$ $(n \geq 3)$
- $SL(n, F)$ and $SL(n, O)$ where F is a number field, and O a ring of algebraic integers $(n \geq 3)$.
- Non-uniform higher rank arithmetic groups.
- Chevalley groups of rank at least 2 over number fields.

Rich rings and semigroups

There are very many prime rings and semigroups (much more then groups):

- $\mathbb{Z}, \mathbb{Q}$, number fields, rings of algebraic integers, free associative algebras over number fields, etc.
- free monoids, various one-relator monoids, etc.

Coordinate algebras in the logical algebraic geometry

As was mentioned above that a successful study of algebraic geometry over a structure G is based on a detailed understanding of the corresponding coordinate algebras.

In the logical algebraic geometry the coordinate algebras over G do exist, but they are Halmos algebras, so algebras in a completely different language than G.

Halmos algebras are not as well studied as groups, so they are not a big help right now.

Non-standard models

Let a structure $\mathbb{A}$ is absolutely interpretable in a structure $\mathbb{B}$ by a code Γ, so $\mathbb{A} \simeq \Gamma(\mathbb{B})$.

In analogy with algebraic groups one may think of the code Γ as a "scheme" that defines $\mathbb{A}$ in $\mathbb{B}$.

In logic there is a notion of *non-standard arithmetic*, which refers to a ring $\tilde{\mathbb{Z}}$ that is first-order equivalent to the "standard" arithmetic $\mathbb{Z}$.

Similarly, non-standard reals is a field $\tilde{\mathbb{R}}$ which is first-order equivalent to the "standard" reals $\mathbb{R}$.

In this vain one can refer to any structure $\mathbb{C}$ which is first-order equivalent to $\mathbb{A}$ as a non-standard model of $\mathbb{A}$. However, we would like to have a notion of a non-standard model of $\mathbb{A}$ relative to the interpretation $\mathbb{A} \simeq \Gamma(\mathbb{B})$.

Non-standard models

Definition

Let $\mathbb{A}$ be a structure that is absolutely (or regularly) interpretable in a structure $\mathbb{B}$, so $\mathbb{A} \simeq \Gamma(\mathbb{B})$ for some code Γ. If $\tilde{\mathbb{B}}$ is a structure which is first-order equivalent to $\mathbb{B}$ then the structure $\Gamma(\tilde{\mathbb{B}})$ is called a non-standard model of $\mathbb{A}$ relative to $\mathbb{B}$.

Non-standard models

Examples

1) A structure $\mathbb{A}$ is trivially interpretable in itself, so in this case the non-standard models are precisely the structures that first-order equivalent to $\mathbb{A}$.
2) Let R be an associative unitary ring. Then the general linear group $GL(n, R)$ is absolutely interpretable in R (as an algebraic group) and every non-standard model of $GL(n, R)$ with respect to R is of the form $GL(n, \tilde{R})$, where $\tilde{R} \equiv R$.
3) A special linear group $SL(n, \mathbb{Z})$ of dimension n over the ring of integers.
4) Let G be a finitely generated torsion-free nilpotent group. In this case a non-standard model of G with respect to $\mathbb{Z}$ is a Hall completion $G^{\tilde{\mathbb{Z}}}$ of G with exponentiation in a non-standard arithmetic $\tilde{\mathbb{Z}}$.

It is worth to mention that many groups have absolute interpretations in $\mathbb{Z}$.

Theorem

Let G be a finitely generated group with decidable word problem. Then G is absolutely interpretable in $\mathbb{Z}$.

Proposition

Let $\mathbb{A} \simeq \Gamma(\mathbb{B})$ be an absolute (or regular) interpretation of $\mathbb{A}$ in $\mathbb{B}$. Then every non-standard model of $\mathbb{A}$ relative to $\mathbb{B}$ is first-order equivalent to $\mathbb{A}$.

Non-standard models

Definition

A scheme $\mathbb{A} \simeq \Gamma(\mathbb{B})$ is called *complete* over $\mathbb{B}$ if any structure which is first-order equivalent to $\mathbb{A}$ is a non-standard model of $\mathbb{A}$ relative to $\mathbb{B}$.

Complete interpretations $\mathbb{A} \simeq \Gamma(\mathbb{B})$ give a reasonable way to solve the first-order classification problem for $\mathbb{A}$ in terms of the non-standard models of $\mathbb{B}$.

Examples:

1) The classical interpretation of $SL(n, \mathbb{Z})$ in $\mathbb{Z}$ (see above) is complete.
2) Malcev's interpretation of a finitely generated torsion-free nilpotent group is not complete.

Non-standard models may provide a lot of interesting properties that are hard to see from the first-order equivalence.

Non-standard models

Theorem

Let G be a torsion-free hyperbolic group and $G \simeq \Gamma_{WP}(\mathbb{Z})$ the interpetation of G in $\mathbb{Z}$ with respect to the word problem in G. Then every non-standard model $\tilde{G} = \Gamma WP(\tilde{\mathbb{Z}})$ where $\tilde{\mathbb{Z}}$ is a non-standard arithmetic satisfy the following properties:

1) The group $\tilde{G}$ is $\tilde{\mathbb{Z}}$-exponential group in the sense of Lyndon, i.e., it admits exponents in the ring $\tilde{\mathbb{Z}}$ with properties similar to the standard exponentiation by $\mathbb{Z}$.

- The group $\tilde{G}$ comes equipped with a metric with values in $\tilde{\mathbb{Z}}$ relative to which it is $\tilde{\mathbb{Z}}$-hyperbolic.

Non-standard models and first-order classification

Theorem

Let a group G be regularly bi-interpretable with $\mathbb{Z}$, $G \simeq \Gamma(\mathbb{Z})$. Then such scheme is complete, i.e., all groups first-order equivalent to G are precisely non-standard models of G, i.e., algebraic groups $\Gamma(\tilde{\mathbb{Z}})$ over a non-standard arithmetic $\tilde{\mathbb{Z}}$.