Anderson modules and abelian extension of fields

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Theorem (Kronecker-Weber). Every finite abelian extension of the field of rational numbers $\mathbf{Q}$ is contained in a cyclotomic extension $\mathbf{Q}(\xi_n)$, $\xi^n = 1$.

This well-known classical result has interesting history.

1. Leopold Kronecker formulated this Theorem in 1853, but the proof of Kronecker was not complete as he himself was aware.
2. Heinrich Weber provided a proof of Kronecker's result in 1886.
3. David Hilbert gave a new proof of Kronecker's original statement in 1896. In his paper D.Hilbert wrote that the Weber's proof of Kronecker's Theorem is correct.
4. In 1909 H.Weber published another proof of Kronecker: Weber, Heinrich, Zur Theorie der zyklischen Zahlkorper, Math. Annalen 67 (1909), 32-60;

Finally, in 1981 (95 years after Weber's paper!!!) Olaf Neumann published the paper:
Neumann, Olaf, Two proofs of the Kronecker-Weber theorem "according to Kronecker, and Weber", J. Reine Angew. Math. 323 (1981), 105-126.
In this paper he found a "gap"in the Weber's first paper (1886). Hence the fist correct proof of the Kronecker-Weber theorem was published by D.Hilbert in 1896.
All details of this history we can find in the paper
Schappacher, Norbert, On the history of Hilbert's twelfth problem: a comedy of errors, Materiaux pour l'histoire des mathematiques au XXe siecle (Nice, 1996), 243- 273, Semin. Congr., 3, Soc. Math. France, Paris, 1998.

And what we can say about maximal abelian extension of another fields? L.Kronecker found other promising case - $k = \mathbf{Q}(\sqrt{-d})$, where d is a natural square free number, an imaginary quadratic field After "proof" of his famous theorem, L. Kronecker described the complex multiplication issue as his "dearest dream of his youth". We note that the Kronecker-Weber theorem may be formulated symbolically: $\mathbf{Q}^{ab} = \mathbf{Q}(exp(2\pi i\mathbf{Q}))$. In the complex multiplication case instead of the exponential function $exp(2\pi iz)$ he wanted to use the other classical function
$j(q) = \frac{1}{q}(1 + 744q + 196884q^2 + ...), q = 2\pi iz$.

Hilbert's original statement of his 12th problem is rather misleading. He seems to imply that the abelian extensions of imaginary quadratic fields are generated by special values of elliptic modular functions $j(q)$, which is not correct. (It is hard to tell exactly what Hilbert was saying, one problem being that he may have been using the term "elliptic function" to mean both the elliptic function $\wp(z)$ (we note that more correctly to write $\wp_L(z)$, since the function $\wp(z)$ depend on the full lattice in $\mathbf{C}$) and the elliptic modular function $j(q)$.) First it is also necessary to use roots of unity, though Hilbert may have implicitly meant to include these. More seriously, while values of elliptic modular functions generate the Hilbert class field, for more general abelian extensions one also needs to use values of elliptic functions $\wp(z)$. For example, the abelian extension $\mathbf{Q}(i, \sqrt[4]{1+2i})/\mathbf{Q}(i)$ is not generated by singular moduli $(j(z), z \in K = \mathbf{Q}(\sqrt{-d}), d > 0,)$ and roots of unity.

One interpretation of Hilbert's twelfth problem asks to provide a suitable analogue of exponential, elliptic, or modular functions, whose special values would generate the maximal abelian extension k^{ab} of a general number field k. In this form, it remains unsolved. But for imaginary quadratic fields this problem was solved in the works of Weber, Hasse, Fuerter, Artin etc..
Let $k = \mathbf{Q}(\sqrt{-d}) \neq \mathbf{Q}, d \in \mathbf{Z}, d > 0$. We define
1) $k^{?} = k(\xi_n | n \in \mathbf{N})-$ cyclotomic extension.
2) $k^{??} = k^{?}(j(\tau) | \tau \in k, Im\tau > 0))$.
Then
3) $k^{ab} = k^{??}(\sqrt{x} | x \in k^{??})$, it is equivalent that $Gal(k^{ab}/k^{??}) = (C_2)^{\infty}$.

This description of k^{ab} is "good" in some sense, but is not in Kronecker spirits. Let give "general" description of K^{ab} in the case $char(K) = 0$ and $\xi_n \in K$ for all n. Hence $K^{ab} = K(\sqrt[n]{x} | x \in K, n \in \mathbf{N})$. It is easy corollary of Kummer theory of cyclic extensions.
The "defect" of this description of k^{ab} is that we apply the functions $\sqrt[n]{x}$ for all elements of $k^{??}$ and not for the elements of k. As was wrote on the previous slide we have to use elliptic function $\wp(z)$. we have:
4) $k^{ab} = k^{??}(\wp(z) | z \in k, Im(z) > 0)$. Finally:
5) $k^{ab} = k(\xi_n, j(z), \wp(z) | z \in k, Im(z) > 0, n \in \mathbf{N})$.

For other finite extensions $\mathbf{Q} < k$ we have only partial results, which show that situation is very difficult and hopeless. The main tools in the study of abelian extensions of k are abelian varieties and modular forms. Recall the result of A.I.Ovseevich [Abilian extensions of the fields of $CM-$type, Funct.Analiz and its Appl., 1974, v. 8, n. 1, 16-24]. Let k be finite totally real extension of $\mathbf{Q}$, $d \in k$ is totally positive (for example, $2-\sqrt{2}$ is totally positive but $1+\sqrt{2} > 2-\sqrt{2}$ is not) and $K = k(\sqrt{-d})$. Then, by definition K is a field of $CM-$type. For exact formulation of A.Ovseevich's results we need many technical definitions. I give "rough"description of K^{ab}, $K-$ a field of $CM-$type obtained by A.Ovseevich. This description is based on very deep results of Shimura-Taniyama[Shimura G., Taniyama J., Complex multiplication of abelian varieties and its application in number theory, Publ. Math. Soc. Japan, 1961.]

A full lattice L in $\mathbf{C}^n$ is a free abelian discrete subgroup of rank $2n$, or $L = \mathbf{Z}e_1 \oplus ... \oplus \mathbf{Z}e_{2n}$, $e_1, ..., e_{2n}$ are linear independent over $\mathbf{R}$. In this case $A = \mathbf{C}^n/L$ is a compact $2n-$dimensional torus. If A is not only analytic variety, but *algebraic variety* we call A an abelian variety. We shall say that A is *polarizable* if it admits a Riemann form: $E : L \times L \to \mathbf{Z}, E(x, y) = -E(y, x), E(ix, iy) = E(x, y)$, $E(iv, v) + iE(v, v) > 0, v \neq 0$.. Then we have: A is an abelian variety iff A is polarizable. An abelian variety A is "associated" with $CM-$field K if an algebra $End_{\mathbf{Z}}A \otimes \mathbf{Q}$ contains a subalgebra K and $[K : \mathbf{Q}] = 2dim_{\mathbf{C}}A = 2n$.

By definition a finite extension $K < L$ is $ST-$extension (Shimura-Taniyama extension) if L is obtained by adding to K coordinates of some points from A of finite order (torsion points). We denote by K^{ST} the union of all $ST-$extensions of K. Shimura and Taniya proved that the extension K^{ST}/K is abelian. If $dimA = 1$ it is an elliptic curve and we know that $K^{ab} = \mathbf{Q}^{ab}K^{ST}$, $K^{ST} = K(\wp(z), \wp'(z)|z \in K)$. Recall that the points $(\wp(z), \wp'(z))$ $\in A$ for any $z \in \mathbf{C}$ and those points have finite order iff $z \in K \subset \mathbf{C}$.

Here we can see the meaning of word "rough" in description of those results. Earlier we formulate the result of Weber and Fuerter: $K^{ab} = K(\xi_n, j(z), \wp(z) |\ z \in K, Im(z) > 0, n \in \mathbf{N})$. It happens that $Gal(K(\wp(z), \wp'(z)|z \in K)/K^{ab})$ is a non-trivial group of exponent two if $K = \mathbf{Q}(\sqrt{-d}, d \in \mathbf{N})$.

Let"s return to the case $dimA > 1$. In general case $K^{ST} \neq K^{ab}$ even if we add the field k^{ab}. Notice that we have very little information about the extension k^{ab}/k, $k-$ totally real field if $k \neq \mathbf{Q}$. A.Ovseevich proved that

(i) $K^{ab} = K^{ST} k^{ab}(\sqrt{x}|x \in I \subseteq K^{ST})$, it is equivalent to the fact: $Gal(K^{ab}/K^{ST} k^{ab})$ is an infinite pro-finite group of exponent two.

(ii) The exists an infinite family $K \subset K_i, i \in \mathbf{N}$ of $CM-$fields, $[K_i : K] = 2$ such that $K^{ab} = K^{ST} k^{ab} S$, where S is a composite of all fields $K^{ab} \cap K_i^{ST}$ and a finite family with those properties does not exist.

Now we will describe the results on abelian extension in characteristic $p > 0$. One of the main persons in this story is Leonard Carlitz (December 26, 1907 - September 17, 1999) was an American mathematician. Carlitz supervised 44 doctorates at Duke University and published over 770 papers.

He began a systematic study of the fields of characteristic $p > 0$. First we recall some analogies between objects in characteristic 0 and $p > 0$. $\mathbf{C}_p$ = completion of $\overline{\mathbf{F}_q((\frac{1}{\theta}))}$

0	$\mathbf{Z}$	$\mathbf{Q}$	$\mathbf{R}$	$\mathbf{C}$	n	$n!$
p	$\mathbf{F}_q[\theta]$	$\mathbf{F}_q(\theta)$	$\mathbf{F}_q((\frac{1}{\theta}))$	$\mathbf{C}_p$	$[n] = \theta^{q^n} - \theta$	$D_n = \prod_{i=1}^{n} [i]^{q^{n-i}}$

0	$exp(x) = \sum_{i=0}^{\infty} \frac{x^n}{n!}$	2π	$i = \sqrt{-1}$
p	$e_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^n}}{D_n}$	$\xi = \prod_{i=1}^{\infty}(1 - \frac{[i]}{[i+1]})$	$\lambda = \sqrt[q-1]{-[1]}$

0	$exp(2\pi in) = 0, n \in \mathbf{Z}$	$exp(sx) = (exp(x))^s$
p	$e_C(\xi\lambda f) = 0, f \in \mathbf{F}_q[\theta]$	$e_C(fz) = C_f(e_C(z)), f \in \mathbf{F}_q[\theta]$

Let $\tau : \mathbf{C}_p \to \mathbf{C}_p, \tau(z) = z^q$, $C_f \in \mathbf{C}_p\{\tau\}$, where $z^q\tau = \tau z$, $C_\theta = \theta\tau^0 + \tau^q$, $C_{fg} = C_f \cdot C_g -$ multiplication in $\mathbf{C}_p\{\tau\}$ and $\phi : \mathbf{F}_q[\theta] \to \mathbf{C}_p\{\tau\}$, $\phi(f) = C_f$ is an injective homomorphism of $\mathbf{F}_q-$algebras.

0	$Gal(\mathbf{Q}(exp(\frac{2\pi i}{n}/\mathbf{Q}) \simeq (\mathbf{Z}/n\mathbf{Z})^*$ is abelian group, $n \in \mathbf{Z}$
p	$Gal(\mathbf{F}_q(\theta)(e_C(\frac{\xi\lambda}{f})/\mathbf{F}_q(\theta)) \simeq (\mathbf{F}_q[\theta]/f\mathbf{F}_q[\theta])^*$ is abel. gr., $f \in \mathbf{F}_q[\theta]$

But, if $\overline{\mathbf{F}_p} = \cup_{i=1}^{\infty}\mathbf{F}_{p^i} -$ algebraic closure of $\mathbf{F}_q$

0	$\mathbf{Q}^{ab} = \mathbf{Q}(exp(\frac{2\pi i}{n}, n \in \mathbf{N})$
p	$\mathbf{F}_q(\theta)^{ab} \neq \overline{\mathbf{F}_p}(\theta)(c_C(\frac{\xi\lambda}{f}),\ f \in \mathbf{F}_q[\theta])$

David Hayes(1937-2011) (one of the Ph.D students of L.Carlitz) and Vladimir Drinfeld independently and simultaneously (in $\sim$ 1974) improved the Carlitz's result.

Let $\mathbf{C}'_p$ be an analogue of the field $\mathbf{C}_p$ where instead of θ we use $\theta' = \theta^{-1}$. Then we have the same analogue of $\mathbf{Q}$: since $\mathbf{F}_q(\theta) = \mathbf{F}_q(\theta')$, but the other analogue of $\mathbf{Z}$: $\mathbf{F}_q[\theta] \neq \mathbf{F}_q[\theta']$ and the other analogue of exponent, π, $\sqrt{-1}$. Let $e'_C(z)$ be an analogue of $e_C(z)$ with respect to θ', then we have $\mathbf{F}_q(\theta)^{ab} = \overline{\mathbf{F}_p}(\theta)(c_C(\frac{\xi\lambda}{f}),\ f \in \mathbf{F}_q[\theta]).\overline{\mathbf{F}_p}(\theta)(c'_C(\frac{\xi'\lambda'}{f}),\ f \in \mathbf{F}_q[\theta'])$.
We note that
$\overline{\mathbf{F}_p}(\theta)(c_C(\frac{\xi\lambda}{f}), f \in \mathbf{F}_q[\theta]) \cap \overline{\mathbf{F}_p}(\theta)(c'_C(\frac{\xi'\lambda'}{f}), f \in \mathbf{F}_q[\theta']) = K \subset \overline{\mathbf{F}_p(\theta)}$, $K \neq \overline{\mathbf{F}_p}(\theta)$.

1. V. G. Drinfeld, "Elliptic modules," Mat. Sb. (N.S.) 94(136) (1974), 594-627.
2. D.Hayes,"EXPLICIT CLASS FIELD THEORY FOR RATIONAL FUNCTION FIELDS", TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, v. 189, 1974,p.77-91.

If to read the paper of D.Hayes we need only basic notions on the field theory, for reading of the paper of V.Drinfeld we need some experience in schemes theory, cohomology theory, algebraic geometry, etc.. Since the main goal of the Drinfeld's paper was contribution to the theory of non-abelian extensions (Langlands program).

Recall that in characteristic 0 case and elliptic curve E (abelian variety of dimension 1) over $\mathbf{Q}$ without complex multiplication corresponding extension of $\mathbf{Q}$, obtained by adding coordinates of torsion points of given order N, has (in general) Galois group $GL_2(\mathbf{Z}/N\mathbf{Z})$. V.Drinfeld in his paper defined an analog of elliptic curve (so called Drinfeld modules) developing ideas of L.Carlitz. Let's return to Carlitz construction of $p-$exponent $e_C(z)$.

Let $\tau : \mathbf{C}_p \to \mathbf{C}_p, \tau(z) = z^q$, $C_f \in \mathbf{C}_p\{\tau\}$, where $z^q\tau = \tau z$, $C_\theta = \theta\tau^0 + \tau^q$, $C_{fg} = C_f \cdot C_g-$ multiplication in $\mathbf{C}_p\{\tau\}$ and $\phi : \mathbf{F}_q[\theta] \to \mathbf{C}_p\{\tau\}$, $\phi(f) = C_f$ is an injective homomorphism of $\mathbf{F}_q-$algebras.
Let $R = \mathbf{F}_q[T]$ then $\mathbf{C}_p$ has many structures of $R-$module as $\mathbf{F}_q-$algebra. For a definition of this structure we need only to have an $\mathbf{F}_q-$homomorphism $\psi : R \to End_{\mathbf{F}_q}(\mathbf{C}_p)$, let call ψ *regular* (or $f-$ *regular*) if $f(x) \in \mathbf{C}_p[x]$ and $\psi(T)(x) = f(x)$. As $\psi(T)(x+y) = \psi(T)(x) + \psi(T)(y)$ hence $f(x) = \sum_{i=0}^r a_i x^{q^i}$. We have bijections between regular $\mathbf{F}_q-$homomorphism $\psi : R \to End_{\mathbf{F}_q}(\mathbf{C}_p)$ and $\mathbf{F}_q-$homomorphism $\phi : R \to \mathbf{C}_p\{\tau\}$, such that to $f-$regular homomorphism ψ the corresponding homomorphism has form $\psi(T) = \sum_{i=0}^r a_i\tau^i$, if $f = \sum_{i=0}^r a_i x^{q^i}$.

The most simple and natural case of $R-$structures on $\mathbf{C}_p$ is when $T \in R = \mathbf{F}_q[T]$ acts on $\mathbf{C}_p$ by multiplication on $\theta : x.T = \theta x$. We will consider the "deformations" of this structure, where $x.T = \theta x + a_1 x^q + a_2 x^{q^2} + ... + a_r x^{q^r},$, $a_r \neq 0$. Thus we received the definition of *DRINFELD* modules. The number r is called *rank* of this module. If $r = 1$ we have *Carlitz* module.

As we note above the notion of *lattice* in $\mathbf{R}$ has analogues in characteristic $p > 0$ as discrete $\mathbf{F}_q[\theta]-$submodule $L \subset \mathbf{R}_q = \mathbf{F}_q((1/\theta))$. We emphasize that the lattices $\mathbf{Z}$ in $\mathbf{R}$ and $\mathbf{F}_q[\theta]$ in $\mathbf{R}_q$ are $co-compact$. As $\overline{\mathbf{R}} = \mathbf{C} = \mathbf{R} \oplus \mathbf{R}i$, then the field $\mathbf{C}$ has co-compact lattice $M = \mathbf{Z}a \oplus \mathbf{Z}b$, where a, b are l.i.over $\mathbf{R}$. It happens that the field $\mathbf{C}_p$ has not co-compact lattice and for every $n \in \mathbf{N}$ there exists a lattice $L = \sum_{j=1}^{m} \oplus \mathbf{F}_q[\theta]a_j \subset \mathbf{C}_p$ of rank m, if $\{a_1, ..., a_m\}$ are l.i. over $\mathbf{R}_q$. L.Carlitz proved that $e_C(\xi\lambda z) = z \prod_{a \in \mathbf{F}_q[\theta]}(1 - \frac{z}{a})$.
It is clear that from this equality we have that $c_C(\xi\lambda a) = 0$ for any $a \in \mathbf{F}$. We can use this fact in order to construct an equation over $\mathbf{F}_q(\theta)$ for $e_C(\frac{\xi\lambda}{f})$, $f \in \mathbf{F}[\theta]$, since $C_f(_C(\frac{\xi\lambda}{f})) = e_C(\xi\lambda) = 0$. Recall that $C_f = \phi(f) \in \mathbf{C}_p\{\tau\}$ and $C_\theta = \theta\tau^0 + \tau$.

By analogy we can define for every lattice $L = \sum_{j=1}^{m} \oplus \mathbf{F}_q[\theta] a_j \subset \mathbf{C}_p$ the corresponding "exponent" function $e_L(z) = z \prod_{a \in L} (1 - \frac{z}{a})$.
For any $f \in \mathbf{F}_q[\theta]$ there exists $C_f \in \mathbf{C}_p[\theta]\{\tau\}$ such that $e_L(fz) = C_f(e_L(z))$. Moreover, $C_{f+g} = C_f + C_g$, $C_{fg} = C_f C_g$, $C_\theta = \theta\tau^0 + \sum_{i=1}^{r} a_i \tau^i$, $a_i \in \mathbf{C}_p$. Hence we have a homomorphism $\phi : \mathbf{F}[\theta] \to End_{\mathbf{F}_q} \mathbf{C}_p\{\tau\}$, $\phi(f) = C_f$, and a structure of $\mathbf{F}_q[T]-$module on $\mathbf{C}_p$, such that $x.T = \theta x + a_1 x^q + a_2 x^{q^2} + ... + a_r x^{q^r} = C_\theta(x)$, as we defined above.

Using those modules V.Drinfeld constructed new algebraic extensions K_L of $K = \mathbf{F}_q(\theta)$ such that for some $f \in \mathbf{F}_q[\theta]$ we have $K_L = K(e_L(a/f) | a \in \mathbf{F}_q[\theta])$. In general case $Gal(K_L/K) \simeq GL_r(\mathbf{F}_q[\theta]/(f))$, where (f) is an ideal generated by f and $r = deg(C_\theta)$ is the rank of the corresponding Drinfeld module. We note that for $r = 1$ we received the Carlitz module and the corresponding fields extension has Galois group $GL_1(\mathbf{F}_q[\theta]/(f)) \simeq (\mathbf{F}_q[\theta]/(f))^*$.

Finally, Anderson, defined a generalization of Drinfeld modules in new, more simple, form. Let $S = \mathbf{C}_p[T, \tau] = \mathbf{C}_p[T]\{\tau\}$ be a non-commutative ring such that $T\tau = \tau T, c^q\tau = \tau c$, $c \in \mathbf{C}_p$, and M be a $S-$module (left). Then M is an *Anderson* module if
(i) M is a free finitely generated as $\mathbf{C}_p[T]-$module and as $\mathbf{C}_p\{\tau\}-$module,
(ii) $T - \theta$ acts on $M/\tau M$ as nilpotent operator.
The dimension n of M over $\mathbf{C}_p\{\tau\}$ is called *dimension* of M and the dimension r over $\mathbf{C}_p[T]$ is called *rank* of M.
Let's understand that Anderson module M is a Drinfeld module iff $n = 1$.

Let $e \in M$ be a generator of M over $\mathbf{C}_p\{\tau\}$, hence any $m \in M$ has a form $m = \sum_{i=0}^{r} a_i \tau^i e$. To define a structure of $\mathbf{C}_p[T]$-module on M we only have to define $Te = \sum_{i=0}^{r} a_i \tau^i e, a_r \neq 0$. In this case the condition (ii) of definition of Anderson module implies that $a_0 = \theta$. It is easy to see that $\{e, \tau e, ..., \tau^{r-1} e\}$ is a basis of M over $\mathbf{C}_p[T]$. Hence the rank of M is r. We consider an example of Anderson module M of dimension $n = 2$ and rank $r = 2$.

Let $\mathbf{e} = (e_1, e_2)$ be a basis of M over $\mathbf{C}_p\{\tau\}$ and $s, \alpha, a, b, c, d \in \mathbf{C}$

$$T\mathbf{e} = \left(\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} -s\alpha & \alpha \\ -s^2\alpha & s\alpha \end{pmatrix} \right) \mathbf{e} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau \mathbf{e},$$

Let denote this Anderson module by $M(s, \alpha, a, b, c, d)$.

We can define a natural definition of morphisms and isomorphisms of Anderson modules. It is not difficult to prove that if $ad \neq bc$, then $M(s, \alpha, a, b, c, d) \simeq C \oplus C$, if $\alpha = 0$, or $M(s, \alpha, a, b, c, d)/C \simeq C$, if $\alpha \neq 0$. In the last case $M(s, \alpha, a, b, c, d)$ is a non-split extension of the two Carlitz modules C.

We note that for Drinfeld module M with $\mathbf{F}_q[T]-$action on $\mathbf{C}_p$: $Tz = \theta z + \sum_{i=1}^{r} a_i z^{q^i}$, the corresponding exponent exp_M may be include in the following commutative diagram

$$\begin{array}{ccc} \mathbf{C}_p & \stackrel{exp_M}{\rightarrow} & \mathbf{C}_p \\ \theta \downarrow & & \downarrow \quad z \mapsto T(z) \\ \mathbf{C}_p & \stackrel{exp_M}{\rightarrow} & \mathbf{C}_p \end{array}$$

It means that exp_M connect the "trivial" $\mathbf{F}_q[T]-$action on $\mathbf{C}_p$: $Tz = \theta z$ with Drinfeld action $Tz = \theta z + \sum_{i=1}^{r} a_i z^{q^i}$. The corresponding lattice $L = \sum_{j=1}^{r} \oplus \mathbf{F}_q[\theta] a_j$ is the kernel of exp_M (above we denoted exp_M by e_L).

For Anderson module M of dimension $n > 1$ we have an analogy diagram

$$\begin{array}{ccc} \mathbf{C}_p^n & \stackrel{exp_M}{\rightarrow} & \mathbf{C}_p^n \\ \theta \downarrow & & \downarrow \quad z \mapsto T(z) \\ \mathbf{C}_p^n & \stackrel{exp_M}{\rightarrow} & \mathbf{C}_p^n \end{array}$$

with corresponding lattice $L = ker(exp_M) \subset \mathbf{C}_p^n$. In the case of Drinfeld module M the map exp_M is surjective, from this fact we get that exp_M defines the action of T. But in general case of Anderson module the map exp_M could be not surjective.

Theorem(Drinfeld) Let M be a Drinfeld module of rank r. Then $h_1(M) = dim_{\mathbf{F}_q[\theta]} ker(exp_M) = r$, and exp_M is surjective.
Theorem(Anderson) Let M be an Anderson module of dimension n and rank r. Then $h_1(M) = dim_{\mathbf{F}_q[\theta]} ker(exp_M) \leq r$. Moreover, exp_M is surjective iff $h_1(M) = r$ and there exist Anderson modules M with $h_1(M) < r$.
Anderson proved that the first group of homologies $H_1(M)$ for Anderson module M and its dimension $h_1(M) = dim_{\mathbf{F}_q[\theta]} H_1(M)$ we can define using functor $Hom_{\mathbf{C}_p[T,\tau]}(M, .)$. Analogously, using functor $M \otimes_{\mathbf{C}_p[T]} .$, we can define $h^1(M) = dim_{\mathbf{F}_q[\theta]} H^1(M)$. Long time was open problem about equality $h_1(M) = h^1(M)$. In some sense it was the problem of isomorphism of the first homologies and cohomologies of Anderson modules.

Theorem(Grishkov, Logachev) There exists Anderson module M with $h_1(M) \neq h^1(M)$. (A.Grishkov, D.Logachev, "$h_1 \neq h^1$ for Anderson $t-$motives," J.Number Theory, 225(2021), 59-89. Doi: 10.1016/j.jnt.2021.01.020)
The counterexample we found between the following $4-$parametric series of Anderson modules $M = M(a, b, c, d)$ of dimension 2 and rank 4 :

$$Te = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \mathbf{e} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau \mathbf{e} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tau^2 \mathbf{e},$$

where $\mathbf{e} = (e_1, e_2)$ is a basis of M over $\mathbf{C}_p\{\tau\}$.
The most simple example of $M = M(a, b, c, d)$ with $h_1(M) \neq h^1(M)$ is $M = M(\theta, \theta^6, \theta^{-2}, 0)$.

It is not difficult to prove that if $\nu_\infty(x) > \frac{q}{q^2-1}$ for all $x \in \{a, b, c, d\}$ then $M = M(a, b, c, d)$ has surjective exp_M and $(h_1(M), h^1(M)) = (4, 4)$. Recall that ν_∞ is a valuation of $\mathbf{C}_p$ such that $\nu_\infty(\theta) = -1$, $\nu_\infty(\alpha) = 0, \alpha \in \mathbf{F}_q^*$ and $\mathbf{C}_p$ is the minimal complete algebraically closed field that contains $\mathbf{F}_q(\theta)$ and $\theta^{-n} \to 0$ if $n \to \infty$. Other simple observation is that $(h_1(M), h^1(M))$ depend only on $(\nu_\infty(a), \nu_\infty(b), \nu_\infty(c), \nu_\infty(d)) \in \mathbf{Q}^4$, in "general" case.

Problem. For given $a, b, c, d \in \mathbf{C}_p$ calculate $(h_1(M), h^1(M))$, where $M = M(a, b, c, d)$.
We (with D.Logachev and S.Ehbauer (UFAM, Brazil)) solved this problem if $a = d = 0$.
Theorem(Grishkov,Esbauer,Logachev) Let $U = R_1 \cup R_2$, where $R_1 = \{(v, u) \in \mathbf{Q}^2 | v + q(u + 1) = 0, u \geq \frac{1}{q-1}\}$,
$R_2 = \{(u, v) \in \mathbf{Q}^2 | u + q(v + 1) = 0, v \geq \frac{1}{q-1}\}$. Then
$h_1(M(0, b, c, 0)) = 4$, if $(v, u) = (\nu(b), \nu(c)) \notin U$, and
$h_1(M(0, b, c, 0)) = 0$ or 4, if $(v, u) \in U$. Moreover, in "general" case $h_1(M(0, b, c, 0)) = 0$ for $(v, u) \in U$.
See Grishkov A.,Eubauer S., Logachev D., "Calculation of h^1 some Anderson $t-$motives. J. of Algebra and its Appl., (2022) 2250017 (31 pages) Doi: 10.1142/S0219498822500177

Conjecture. There exists a subset U of $\mathbf{Q}^4$ which is the complement to a union of countably many (maybe even finitely many - we do not know) linear subspaces of dimension ≤ 3 such that if $(\nu_\infty(a), \nu_\infty(b), \nu_\infty(c), \nu_\infty(d)) \in \mathbf{Q}^4 \setminus U$, then $h_1(M(A)) = 4$.

The other important problem in the theory of Anderson modules is to understand the correspondence between the set $\mathbf{M}_n$ of all Anderson modules M such that $h_1(M) = r$ =rank of M, and the set $\mathbf{L}_n$ of all finite dimensional lattice in $\mathbf{C}_p^n$. The main conjecture is
Conjecture. Let $\phi_n : \mathbf{M} \to \mathbf{L}$ be a map such that $\phi_n(M) = ker(exp_M)$, $M \in \mathbf{M}_n$. Is an image $\phi_n(\mathbf{M}_n) \subseteq \mathbf{L}_n$ open? Is a set $\phi_n^{-1}(L)$ finite? Here $L \in \mathbf{L}_n$ is a "generic" in some sense.

Let $M = C \oplus ... \oplus C$ be a direct sum of n copies of Carlitz module C and L be the corresponding lattice. We proved that there exist some neighborhoods V of M and W of L such that $\phi : V \to W$ is a bijection. See Grishkov A.; Logachev D. "Lattice map for Anderson t-motives: First approach". J. of Number Theory, v. 180, p. 373-402, 2017. 10.1016/j.jnt.2017.04.004

Notice that one of "motives" of introduction of Anderson modules was the fact that some natural operations with Drinfeld module give the modules which are not Drinfeld modules.
Consider two Drinfeld modules M_1 and M_2 of ranks r_1 and r_2 correspondingly. It means that M_1 and M_2 are free $\mathbf{C}_p[T]-$modules of dimension r_1 and r_2. Then $M = M_1 \otimes_{\mathbf{C}_p[T]} M_2$ is $\mathbf{C}_p[T]-$modules of dimension $r = r_1 r_2$. We can define a structure of left $\mathbf{C}_p\{\tau\}-$module on M such that $\tau(v \otimes w) = \tau v \otimes \tau w$. Anderson proved that M is an Anderson module of dimension $n = n_2 r_1 + n_1 r_2$ and rank $r_1 r_2$.

For example, let C_1 and C_2 be two copies of Carlitz module with bases e_1 and e_2. Then $M = C_1 \otimes_{\mathbf{C}_p[T]} C_2$ is an Anderson module of rank $r = 1$, since M as $\mathbf{C}_p[T]-$module has a basis $e_1 \otimes e_2$. The dimension of M over $\mathbf{C}_p\{\tau\}$ is $n = 2$. A basis of M over $\mathbf{C}_p\{\tau\}$ is $e_1 \otimes e_2, e_1 \otimes \tau e_2$. Indeed, we have $Te_1 \otimes e_2 = e_1 \otimes Te_2$, since tensor product we have over $\mathbf{C}_p[T]$. Hence
$Te_1 \otimes e_2 = (\theta e_1 + \tau e_1) \otimes e_2 = \theta(e_1 \otimes e_2) + \tau e_1 \otimes e_2 = e_1 \otimes Te_2 = e_1 \otimes (\theta e_2 + \tau e_2) = \theta(e_1 \otimes e_2) + e_1 \otimes \tau e_2$. Then
$\tau e_1 \otimes e_2 = e_1 \otimes \tau e_2$. Analogously we get
$\tau^2 e_1 \otimes e_2 = e_1 \otimes \tau^2 e_2 = \tau e_1 \otimes \tau e_2 = \tau(e_1 \otimes e_2)$, etc.. Note that the action of $T - \theta$ on $M/\tau M$ is not $= 0$.

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