Rich structures and weak second order logic

Alexei Miasnikov
(Stevens Institute)

Omsk
November 18th 2021

Outline

The main question I will focus on in this talk is:

How expressive is first-order logic (FO) in a given group (ring, semigroup, etc)?

Measuring expressive power of FO

Let $\mathbb{A}$ be a structure.

How to measure expressive power of the first-order logic in $\mathbb{A}$?

For example, one may look at the following questions:

- Try to describe $\mathbb{A}$, say up to isomorphism, by FO sentences that are true in $\mathbb{A}$.
- Try to see if the natural algebraic objects associated with $\mathbb{A}$ are FO definable (interpretable) in $\mathbb{A}$.

FO rigidity and QFA

Let $\mathbb{A}$ be finitely generated.

- First-order rigidity: for any f.g. $\mathbb{B}$ one has

$$\mathbb{A} \equiv \mathbb{B} \Longrightarrow \mathbb{A} \simeq \mathbb{B}$$

- QFA: there is a FO sentence $\phi_{\mathbb{A}}$ such that for any f.g. $\mathbb{B}$ one has

$$\mathbb{B} \models \phi_{\mathbb{A}} \Longleftrightarrow \mathbb{A} \simeq \mathbb{B}$$

Malcev problems

General Malcev problem: describe definable subgroups in a given group.

In fact, Malcev asked about definable subgroups only in free groups, but he studied this question and used some results in his study of Tarski problems in

- free nilpotent groups
- free solvable groups
- the classical matrix groups

Malcev problems

Theorem [Khalampovich - M., Perin - Pillay - Sklinos -Tent]

Only subgroups $1, F$, and all cyclics, are definable in a free non-abelian group F. The same holds for torsion-free hyperbolic groups.

Proofs are rather difficult, based on the techniques developed in solutions to Tarski problems.

Finitely generated abelian groups

Only finite and finite index subgroups are definable in finitely generated abelian groups.

With Sasha Treyer we looked at definable subgroups in finitely generated nilpotent groups.

Question: What are groups where all subgroups are definable?

1. finite groups
2. infinite cyclic groups ($\mathbb{Z}$)
3. what else?

Strange question

Question: What are groups where all subgroups are definable?

1. finite groups
2. infinite cyclic groups ($\mathbb{Z}$)
3. what else?

Uniform definability of subgroups

Even a stronger condition:

Subgroups in a group G are uniformly definable if for any $n \in \mathbb{N}$ there exists a first order formula $\phi_n(x_1, \ldots, x_n, y)$ such that for any $a_1, \ldots, a_n, b \in G$

$$G \models \phi_n(a_1, \ldots, a_n, b) \iff b \in \langle a_1, \ldots, a_n, \rangle$$

Question: are there infinite groups with uniformly definable subgroups?

Not free, not torsion-free hyperbolic, not $\mathbb{Z}$, not finitely generated abelian.

Uniform definability of subgroups

Even a stronger condition:

Subgroups in a group G are uniformly definable if for any $n \in \mathbb{N}$ there exists a first order formula $\phi_n(x_1, \ldots, x_n, y)$ such that for any $a_1, \ldots, a_n, b \in G$

$$G \models \phi_n(a_1, \ldots, a_n, b) \iff b \in \langle a_1, \ldots, a_n, \rangle$$

Question: are there infinite groups with uniformly definable subgroups?

Not free, not torsion-free hyperbolic, not $\mathbb{Z}$, not finitely generated abelian.

Surprise

Theorems

The following groups have uniformly definable subgroups:

1. (based on Khelif) $BS(1, m)$, $m > 1$ and $\mathbb{Z}_n wr \mathbb{Z}$, $n > 1$.
2. Free metabelian of finite rank > 1.
3. (based on Khelif) The Thompson group F.
4. (M. - Sohrabi) $SL(n, \mathbb{Z})$,$SL(n, \mathbb{Q})$, $SL(n, F)$, $SL(n, O)$ where F is a number field, and O a ring of algebraic integers ($n \geq 3$)
5. (based on Avni, Lubotzky, Meiri) Non-uniform higher rank arithmetic groups.
6. (based on Segal and Tent) Chevalley groups of rank at least 2 over number fields.
7. (based on Macintyre and Trofimov) Algebraically closed groups.

Surprise

Theorems

The following groups have uniformly definable subgroups:

1. (based on Khelif) $BS(1, m)$, $m > 1$ and $\mathbb{Z}_n wr \mathbb{Z}$, $n > 1$.
2. Free metabelian of finite rank > 1.
3. (based on Khelif) The Thompson group F.
4. (M. - Sohrabi) $SL(n, \mathbb{Z})$,$SL(n, \mathbb{Q})$, $SL(n, F)$, $SL(n, O)$ where F is a number field, and O a ring of algebraic integers ($n \geq 3$)
5. (based on Avni, Lubotzky, Meiri) Non-uniform higher rank arithmetic groups.
6. (based on Segal and Tent) Chevalley groups of rank at least 2 over number fields.
7. (based on Macintyre and Trofimov) Algebraically closed groups.

Even more surprise

It turns out all these groups G satisfy many other uniform definability properties for n-generated subgroups:

For any n there exists a formula $\phi_n(x_1, \ldots, x_n)$ such that for any $a_1, \ldots, a_n \in G$ the formula $\phi_n(a_1, \ldots, a_n)$ holds in G iff the subgroup generated by $a_1, \ldots, a_n$ is

1. free (or nilpotent of class c, or solvable of class c, ...)
2. finitely presented
3. residually finite
4. the word problem is decidable (in polynomial time).
5. the conjugacy problem is decidable (in polynomial time).

Clearly, something very general (beyond group theory) is going on here.

Expressive power of the first-order logic in such groups

Obviously, the expressive power of the first-order logic in such groups is great.

Two questions:

- How great is it?
- Where does it come from?

In the rest of the talk I will try to answer these questions.

The weak second order logic

The natural answer to the first question (how powerful is the first-order logic over G?) might be in the form:

The first-order logic over G is as expressive as some more powerful logic $\mathcal{L}$ over G.

What could be such $\mathcal{L}$: the second order logic? or $L_{\omega_1,\omega}$?

Seems unreasonably powerful.

Quick question 1

Is there an infinite group where the first order logic has the same expressive power as $L_{\omega_1,\omega}$? Or the second order logic?

The weak second order logic

The natural answer to the first question (how powerful is the first-order logic over G?) might be in the form:

The first-order logic over G is as expressive as some more powerful logic $\mathcal{L}$ over G.

What could be such $\mathcal{L}$: the second order logic? or $L_{\omega_1,\omega}$?

Seems unreasonably powerful.

Quick question 1

Is there an infinite group where the first order logic has the same expressive power as $L_{\omega_1,\omega}$? Or the second order logic?

Rich structures

The Weak Second Order Logic would be perfect.

The main (intuitive) definition

An algebraic structure $\mathbb{A}$ is rich if the the first-order logic over $\mathbb{A}$ has the same expressive power as the weak second order logic over $\mathbb{A}$.

Now I need to make this definition more precise.

Hereditary finite superstructures

For a group $\mathbb{A} = \langle A; \cdot, 1\rangle$ define a superstructure $HF(\mathbb{A})$ as follows (below $Pf(X)$ is the set of all finite subsets of a set X).

Definition of hereditary finite sets over A

- $HF_0(A) = A$,
- $HF_{n+1}(A) = HF_n(A) \cup Pf(HF_n(A))$,
- $HF(A) = \bigcup_{n\in\omega} HF_n(A)$.

Put

$$HF(\mathbb{A}) = \langle HF(A); P_A, \cdot, 1, \in\rangle,$$

where P_A is the predicate defining A in $HF(A)$, $\cdot, 1$ are defined on A, as in the group $\mathbb{A}$, and $\in$ is the membership predicate on $HF(A)$.

Simillary, one defines $HF(\mathbb{A})$ for any structure $\mathbb{A}$.

For a group $\mathbb{A} = \langle A; \cdot, 1\rangle$ define a superstructure $HF(\mathbb{A})$ as follows (below $Pf(X)$ is the set of all finite subsets of a set X).

Definition of hereditary finite sets over A

- $HF_0(A) = A$,
- $HF_{n+1}(A) = HF_n(A) \cup Pf(HF_n(A))$,
- $HF(A) = \bigcup_{n\in\omega} HF_n(A)$.

Put

$$HF(\mathbb{A}) = \langle HF(A); P_A, \cdot, 1, \in\rangle,$$

where P_A is the predicate defining A in $HF(A)$, $\cdot, 1$ are defined on A, as in the group $\mathbb{A}$, and $\in$ is the membership predicate on $HF(A)$.

Simillary, one defines $HF(\mathbb{A})$ for any structure $\mathbb{A}$.

Hereditary finite superstructures

For a group $\mathbb{A} = \langle A; \cdot, 1\rangle$ define a superstructure $HF(\mathbb{A})$ as follows (below $Pf(X)$ is the set of all finite subsets of a set X).

Definition of hereditary finite sets over A

- $HF_0(A) = A$,
- $HF_{n+1}(A) = HF_n(A) \cup Pf(HF_n(A))$,
- $HF(A) = \bigcup_{n\in\omega} HF_n(A)$.

Put

$$HF(\mathbb{A}) = \langle HF(A); P_A, \cdot, 1, \in\rangle,$$

where P_A is the predicate defining A in $HF(A)$, $\cdot, 1$ are defined on A, as in the group $\mathbb{A}$, and $\in$ is the membership predicate on $HF(A)$.

Simillary, one defines $HF(\mathbb{A})$ for any structure $\mathbb{A}$.

The weak second-order theory

The main point:

The first-order theory of $HF(\mathbb{A})$ has the same expressiveness as the *weak second order theory* of $\mathbb{A}$.

Indeed, one can view elements from $HF(\mathbb{A})$ as of two sorts: ones are from A, and the others are from $HF(A) \smallsetminus A$. So one can use formulas of the second order logic where variables $x_1, x_2, \ldots,$ run over A and variables $Y_1, Y_2, \ldots$ run over sets from $HF(A)$.

This logic is extremely powerful: all the "finite math" is there (arithmetic, typical finite constructions, etc.)

Rich structures

Definition of rich structures

A structure $\mathbb{A}$ is rich if for every first-order formula $\phi(x_1, \ldots, x_n)$ in the language of $HF(\mathbb{A})$ there is a first-order formula $\phi^*(x_1, \ldots, x_n)$ in the language of $\mathbb{A}$ such that for any assignment of variables $x_1 \to a_1, \ldots, x_n \to a_n$, where $a_i \in \mathbb{A}$, one has

$$HF(\mathbb{A}) \models \phi(a_1, \ldots, a_n) \Longleftrightarrow \mathbb{A} \models \phi^*(a_1, \ldots, a_n).$$

Sometimes we allow to extend the language of $\mathbb{A}$ by finitely many constants (but usually it is not needed).

Another look at the the weak second-order logic

Now we describe WSO logic as a fragment $L^{ar}_{\omega_1,\omega}$ of $L_{\omega_1,\omega}$.

We start with the first-order logic L_{FO} and build it up to $L^{ar}_{\omega_1,\omega}$ adding some formulas ϕ from $L_{\omega_1,\omega}$ and simultaneously defining two things:

- extend the standard enumeration of formulas in L_{FO} to the new formulas $\phi \in L^{ar}_{\omega_1,\omega}$;
- for each such ϕ define its complexity.

By ϕ_i we denote the formula from $L^{ar}_{\omega_1,\omega}$ with index i in our enumeration.

In building $L^{ar}_{\omega_1,\omega}$ we use $\vee, \wedge, \neg, \forall, \exists$ as usual.

Now we explain how to use infinite disjunctions and conjunctions.

Logic $L^{ar}_{\omega_1,\omega}$

Let

$$\Phi = \{\phi_i \mid i \in I\}$$

be a set of formulas from $L^{ar}_{\omega_1,\omega}$ such that

- the set of indices I is an arithmetic subset of $\mathbb{N}$,
- all $\phi_i, i \in I$, have uniformly bounded complexity, say $\leq r$,
- all free variables of $\phi_i, i \in I$, are among $x_1, \ldots, x_m$ for some m.

Then

$$\bigwedge\{\phi_i \mid i \in I\} \;\; and \;\; \bigvee\{\phi_i \mid i \in I\}$$

are formulas from $L^{ar}_{\omega_1,\omega}$.

If $I \in \Pi_s$ then the complexity of these formulas $\leq \max\{r, s\} + 1$.

Weak Second Order Logic

Theorem [Belyaev-Taiclin]

The following holds for every algebraic structure $\mathbb{A}$ of language L:

1) for every formula $\phi(x_1, \ldots, x_n) \in L^{ar}_{\omega_1,\omega}$ there is a formula $\phi^*(x_1, \ldots, x_n)$ in the language of $HF(\mathbb{A})$ such that for any assignment of variables $x_1 \to a_1, \ldots, x_n \to a_n$, where $a_i \in A$ one has

$$\mathbb{A} \models \phi(a_1, \ldots, a_n) \Longleftrightarrow HF(\mathbb{A}) \models \phi^*(a_1, \ldots, a_n).$$

2) for every formula $\psi(x_1, \ldots, x_n)$ in the language of $HF(\mathbb{A})$ there is a formula $\psi^*(x_1, \ldots, x_n) \in L^{ar}_{\omega_1,\omega}$ such that for any assignment of variables $x_1 \to a_1, \ldots, x_n \to a_n$, where $a_i \in A$, one has

$$HF(MA) \models \psi(a_1, \ldots, a_n) \Longleftrightarrow \mathbb{A} \models \psi^*(a_1, \ldots, a_n).$$

Example

Let G be a rich group.

For a given number $n \in \mathbb{N}$ consider an effective enumeration of all group words in variables $x_1, \ldots, x_n$, say

$$w_0(x_1, \ldots, x_n), w_1(x_1, \ldots, x_n), \ldots$$

Then the formula

$$\Psi_n(x_1, \ldots, x_n, y) = \bigvee_{i \in \mathbb{N}} y = w_i(x_1, \ldots, x_n)$$

belongs to $L^{ar}_{\omega_1,\omega}$ (when L is the group language) and uniformly defines in G all n-generated subgroups. Indeed, for any $a_1, \ldots, a_n, b \in G$ one has

$$G \models \Psi_n(a_1, \ldots, a_n, b) \Longleftrightarrow b \in \langle a_1, \ldots, a_n \rangle$$

Rich groups

Looking again at the groups we discussed above.

The following groups are rich:

- $BS(1, m)$, $m > 1$ and $\mathbb{Z}_n wr \mathbb{Z}$, $n > 1$.
- Free metabelian of finite rank > 1.
- Thompson group F.
- $SL(n, \mathbb{Z})$ and $SL(n, \mathbb{Q})$ ($n \geq 3$)
- $SL(n, F)$ and $SL(n, O)$ where F is a number field, and O a ring of algebraic integers ($n \geq 3$).
- Non-uniform higher rank arithmetic groups.
- Chevalley groups of rank at least 2 over number fields.
- Algebraically closed groups.

This explains why there is unifom definablility of subgroups there.

Rich rings and semigroups

There are very many rich rings and semigroups (much more then groups):

- $\mathbb{Z}, \mathbb{Q}$, number fields, rings of algebraic integers, free associative algebras, etc.
- free monoids, various one-relator monoids, etc.

How to prove richness of a structure

Now I will try to explain how one can prove that a given structure is rich.

For this I need a notion of bi-interpretability of structures.

Bi-interpretability

Let $\mathbb{A}$ and $\mathbb{B}$ be two structures, possibly of different signatures.

Suppose that $\mathbb{B}$ is interpeted in $\mathbb{A}$ as $\mathbb{B}^*$ and $\mathbb{A}$ is interpreted in $\mathbb{B}$ as $\mathbb{A}^*$ such that

- for interpretations

$$\mathbb{A} \succ \mathbb{B}^* \simeq \mathbb{B} \succ \mathbb{A}^*$$

  there is an isomorphism $\lambda : \mathbb{A} \to \mathbb{A}^*$ definable in $\mathbb{A}$.

- for interpretations

$$\mathbb{B} \succ \mathbb{A}^* \simeq \mathbb{A} \succ \mathbb{B}^*$$

  there is an isomorphism $\mu : \mathbb{B} \to \mathbb{B}^*$ definable in $\mathbb{B}$

In this case we say $\mathbb{A}$ and $\mathbb{B}$ are bi-interpretable in each other.

Bi-interpretability: picture

Observe, that $UT_3(\mathbb{Z})$ and the arithmetic $\mathbb{Z}$ are mutually interpretable in each other but they are not bi-interpretable in each other.

Theorem [Kharlampovich-M.-Sohrabi]

Let G be a finitely generated nilpotent group. Then G is not bi-interpretable with $\mathbb{Z}$.

Where the richness come from?

The following results show where the richness comes from.

Main technical theorem

Let $\mathbb{A}$ be a structure.

- If $\mathbb{A}$ is bi-interpretable with $HF(\mathbb{A})$ then $\mathbb{A}$ is rich.
- If $\mathbb{A}$ is bi-interpretable with $HF(\mathbb{B})$ for some structure $\mathbb{B}$ then $\mathbb{A}$ is rich.
- If $\mathbb{A}$ is bi-interpretable with a rich structure $\mathbb{B}$ then $\mathbb{A}$ is rich.

Corollary

If a structure $\mathbb{A}$ is bi-interpretable with arithmetic $\mathbb{N}$, or $\mathbb{Z}$, or $\mathbb{Q}$, then $\mathbb{A}$ is rich.

Theorem [Kharlampovich-M.- Sohrabi]

Let G be a free metabelian group of finite rank $r \geq 2$. Then G is bi-interpretable with $\mathbb{Z}$. Hence G is rich.

The case $r = 2$ was done be Khelif. The result is not easy.

Corollaries:

- Finitely generated subgroups of G are uniformly definable,
- Finitely presented subgroups of G are uniformly definable,
- The set of bases is definable,
- G is QFA, prime and homogeneous.
- ...

A description of finitely generated commutative unitary rings bi-interpretable with $\mathbb{N}$ was given by Aschenbrenner, Khelif, Naziazeno, Scanlon in 2020.

Corollary

Every infinite finitely generated integral domain A is bi-interpretable with $\mathbb{N}$. Hence A is rich.

Rich fields

Theorem [Dittman and Pop, 2020]

Let K be an infinite finitely generated field. If char(K)=2 and $dim(K) > 3$ assume that resolution of singularities above $\mathbb{F}_2$ holds. Then K is bi-interpretable with $\mathbb{Z}$. Hence K is rich.

Group algebras over free groups

For a group G and a field K by $K(G)$ we denote the group algbera of G over K.

Theorem [Kharlampovich-M.], 2017

Let F be a free non-abelian group and K a field. Then $K(F)$ and $HF(K)$ are bi-interpretable in each other. In particular, the ring $K(F)$ is rich.

Furthermore, $HF(F)$ is interpretable in $K(F)$. The ring $K(F)$ knows much more about F then the group F itself!

For a set X and a field K by $K[X]$ we denote the free associative algebra with basis X and coefficients in K.

Theorem [Kharlampovich-M.], 2016

Let X be a finite set with $|X| > 1$ and K a field. Then $K[X]$ and $HF(K)$ are bi-interpretable in each other. In particular, the ring $K[X]$ is rich.

The list superstructure

To prove the results such as above it is convenient to use the *list superstructure* $S(\mathbb{A}, \mathbb{N})$ over a structure $\mathbb{A}$ instead of $HF(\mathbb{A})$, whose first-order theory has the same expressive power as of $HF(\mathbb{A})$.

Let $\mathbb{A}$ be a structure. Put

$$S(\mathbb{A}, \mathbb{N}) = \langle \mathbb{A}, \mathbb{N}, S(A); \frown, \ell(s), t(s, i, a) \rangle,$$

where:

$\mathbb{N} = \langle N, +, \cdot, 0, 1 \rangle$ is the standard arithmetic,

$S(A)$ is the set of all lists over $\mathbb{A}$,

$\frown$ is concatenation of lists,

$\ell : S(A) \to N$ is the length function,

$S(\mathbb{A}, \mathbb{N}) \models t(s, i, a) \iff s = (s_1, \ldots, s_n) \in S(A), i \in N$, and $a = s_i \in A$.

A great effort of several mathematicians: Scott, Newman, Cohn, Wheeler, Macintyre, Bokut, Taitslin, Cherlin, Trofimov, Belegradek, Belyaev - led to a remarkable study of existentially closed groups, rings and semigroups.

A powerful unifying approach in this area was developed by Belyaev and Taitslin.

Theorems

Existentially closed objects in the following classes of structures: groups, torsion-free groups, semigroups, semigroups with cancellation, inverse semigroups, skew fileds, associative rings - are rich.

To prove these results one needs superstructure of finite binary predicates over $\mathbb{A}$.
Let $FBP(A)$ we denote the set of all finite binary predicates over A.

Put

$$FBP(\mathbb{A}) = \langle \mathbb{A}, FBP(A); s(x, y, z)\rangle$$

where x and y run over A, z runs over $FBP(A)$ and $s(a, b, H)$ holds in $FBP(\mathbb{A})$ if and only if $(a, b) \in H$.

Theorem

Let $\mathbb{A}$ be an infinite structure. Then $HF(\mathbb{A})$ and $FBP(\mathbb{A})$ are absolutely bi-interpretable in each other.

Note, that every non-trivial algebraically closed groups is also existentially closed.

Corollary

Every non-trivial algebraically closed groups is rich.

Corollary

Every countable groups embeds into a countable (algebraically closed, simple) rich group.

This follows from the properties of algebraically closed groups.

Countably infinite groups (with a fixed underlying set) form a Polish space $\mathcal{G}$ with a suitable metric.

A group property $\mathcal{P}$ is called *generic* if the subset of groups in $\mathcal{G}$ satisfying $\mathcal{P}$ is comeager in $\mathcal{G}$.

Also, in this case we say that a generic infinite countable group satisfies $\mathcal{P}$.

Theorem [Elekes, Geher, Kanalas, Katay, and Keleti]

A generic infinite countable group is algebraically closed.

Corollary

A generic infinite countable group is rich.

QFA and rich structures

Many of known results on QFA of groups and rings come from the following result.

Theorem [Kelif, Nies

Let $\mathbb{A}$ be a finitely generated structure in a finite signature. If $\mathbb{A}$ is bi-interpretable with $\mathbb{Z}$ then $\mathbb{A}$ is QFA.

In fact, a more general result holds.

Theorem [K-M-S]

Let $\mathbb{A}$ be a rich structure in a finite signature. If $\mathbb{A}$ is generated by a finite set S with an arithmetic diagram $D_S(\mathbb{A})$ then $\mathbb{A}$ is QFA.

Primality and homogeneity

Theorem [Oger-Sabbagh]

If an infinite structure $\mathbb{A}$ is bi-interpretable with $\mathbb{Z}$ then $\mathbb{A}$ is prime in $Th(\mathbb{A})$ and homogeneous.

Quick question 2

Let G be a finitely generated group with arithmetic multiplication table. Is it true that only arithmetic types can be realized in G?

Problem [B.Plotkin]

Let A and B two finitely generated groups which realize the same types (isotypic). Is it true that A and B are isomorphic?

Primality and homogeneity

Theorem [Oger-Sabbagh]

If an infinite structure $\mathbb{A}$ is bi-interpretable with $\mathbb{Z}$ then $\mathbb{A}$ is prime in $Th(\mathbb{A})$ and homogeneous.

Quick question 2

Let G be a finitely generated group with arithmetic multiplication table. Is it true that only arithmetic types can be realized in G?

Problem [B.Plotkin]

Let A and B two finitely generated groups which realize the same types (isotypic). Is it true that A and B are isomorphic?

Theorem [Oger-Sabbagh]

If an infinite structure $\mathbb{A}$ is bi-interpretable with $\mathbb{Z}$ then $\mathbb{A}$ is prime in $Th(\mathbb{A})$ and homogeneous.

Quick question 2

Let G be a finitely generated group with arithmetic multiplication table. Is it true that only arithmetic types can be realized in G?

Problem [B.Plotkin]

Let A and B two finitely generated groups which realize the same types (isotypic). Is it true that A and B are isomorphic?

In different algebraic structures $\mathbb{A}$ the weak second order logic naturally occurs in different forms:

- HF-logic over $\mathbb{A}$ (via superstructure $HF(\mathbb{A})$),
- Logic of lists over $\mathbb{A}$ (via the list superstructure $S(\mathbb{A}, \mathbb{N})$),
- Logic of finite (binary) predicates over $\mathbb{A}$ (via the superstructure $FBP(\mathbb{A})$),
- Logic $L^{ar}_{\omega_1,\omega}$ of infinite arithmetic conjunctions and disjunctions,
- Logic $L^{rec}_{\omega_1,\omega}$ of recursive conjunctions and disjunctions.

Theorem Belyaev-M.

All the logics mentioned above are effectively equivalent over any (infinite) structure $\mathbb{A}$, i.e., given a formula ϕ in one logic one can effectively find a formula ϕ^* in another logic which is equivalent to ϕ over $\mathbb{A}$.

The weak second order logic

Each of these forms of WSO is especially convenient in solving a certain type of problems.

However, it would be useful to have a logic where you can use all these tools simultaneously.

Recently jointly with Belyaev we developed such a "universal weak second order logic" L_U, which naturally contains all the logics mentioned above.

Theorem

Let $\mathbb{A}$ be an infinite structure of finite signature such that $\mathbb{A}$ is bi-interpretable with $HF(\mathbb{A})$. Then for every formula ϕ of L_U over $\mathbb{A}$ one can effectively find a first-order formula ϕ^* in the language of $\mathbb{A}$ which is equivalent to ϕ over $\mathbb{A}$.