Structures with Non-Standard Kleene Stars

Omsk Algebraic Webinar

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Stepan Kuznetsov

Steklov Mathematical Institute of the RAS, Moscow

Kleene Star

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- This is due to its inductive nature: even in purely “algebraic” theories, like equational ones, it exhibits behaviour which makes it close to arithmetic.

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- On formal languages, naturally $B \cdot C = \{uv \mid u \in B, v \in C\}$ and $\preceq$ is $\subseteq$.

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- With these operations, the equational theory (that is, the problem of equivalence of regular expressions) is decidable and belongs to PSPACE.
- However, the first-order theory and even the Horn theory (quasi-equations) is Π_1^1-complete (!).
- Therefore, we shall consider only equational theories.

Relational Algebra

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- The unit is the diagonal relation: $\mathbf{1} = \{(x, x) \mid x \in W\}$.
- Kleene star, again, is defined as the supremum of n-th powers, and gives the *reflexive-transitive closure:*

$$R^* = \mathbf{1} \cup R \cup (R \circ R) \cup (R \circ R \circ R) \cup \ldots$$

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  - Add two constants, a and d, and consider the following theory: $\mathsf{KA} + \{a^* \succ d, \mathbf{1} \preceq d, a \preceq d, a^2 \preceq d, \ldots\}$
- However, as shown by Kozen, the equational theory for general Kleene algebras is the same.

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- In the presence of residuals, the equational theory of *-continuous structures becomes Π^0_1-complete (Buszkowski & Palka 2007).
- Therefore, it cannot coincide with the equational theory of all structures, since it is recursively enumerable (the fixed point conditions are represented by a finite axiomatization).

Let us give the formal definition of residuated Kleene lattices (RKL).

An RKL is a partially ordered algebraic structure

$$\mathscr{A} = (A, \preceq, \cdot, \mathbf{1}, \mathbf{0}, \vee, \wedge, \setminus, /, {}^*),$$

where:

1. $(A, \preceq, \vee, \wedge)$ is a lattice with $\mathbf{0}$ being its smallest element;
2. $(A, \cdot, \mathbf{1})$ is a monoid;
3. $\setminus$ and $/$ are residuals of $\cdot$ w.r.t. $\preceq$:

   $$a \setminus c = \max_{\preceq} \{b \mid a \cdot b \preceq c\}, \qquad c \,/\, b = \max_{\preceq} \{a \mid a \cdot b \preceq c\}$$

   (these maxima are postulated to exist);
4. $a^* = \min_{\preceq} \{b \mid \mathbf{1} \preceq b \text{ and } a \cdot b \preceq b\}$ (such minima are also postulated to exist).

Residuated Kleene Lattices

The following properties follow from the definition:

- $\cdot$ is monotone w.r.t. $\preceq$; $\backslash$ and $/$ are monotone by the numerator and anti-monotone by the denominator (Lambek 1958);
- $\mathbf{0}$ is the zero w.r.t. $\cdot$;
- the left Kleene star is also the right one: $a^* = \min_{\preceq} \{b \mid \mathbf{1} \preceq b \text{ and } b \cdot a \preceq b\}$;
- *Pratt's normality theorem:* if there exists $\sup_{\preceq} \{a^n \mid n \in \mathbb{N}\}$, then it coincides with a^*.

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An RKL is **-continuous,* if these suprema exist for all $a \in A$.

ACT and $\mathbf{ACT}_\omega$

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- $\mathbf{ACT}_\omega$ is nice.
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  - Standard natural interpretations (relational and language models) are *-continuous, thus, they are models for $\mathbf{ACT}_\omega$.
  - There are issues with completeness, but they are resolved in syntactic concept lattice (SCL) semantics by Wurm (2017), different versions of which generalize language and relational models.

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- However, standard semantics (relational, language, SCL) are useless if we wish to pinpoint *differences* between **ACT** and **ACT**$_\omega$.
- **We need non-standard Kleene stars.**
- We show two concrete cases of using non-standard algebraic, which are *ad hoc* inventions.

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- However, in the presence of divisions the following *Pratt's normality theorem* holds: if the supremum for a^* exists, then it should be a^*.
- Thus, we should avoid such suprema.
- In particular, if the RKL is complete (as a lattice), it should be *-continuous.

The First Example of a Non-*-Continuous RKL

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- Multiplication in the middle part is componentwise addition: $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + b_1, a_2 + b_2)$; $\bot \cdot x = x \cdot \bot = \bot$ and $\top \cdot y = y \cdot \top = \top$ (for $y \neq \bot$).

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- Kleene star: $\bot^* = \bot$, $(0, 0)^* = (0, 0)$ and for $x \succ (0, 0)$ we have $x^* = \top$.

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Residuals:

- $x \,/\, \bot = \top$, $\top \,/\, x = \top$, $\bot \,/\, y = \bot$, $y \,/\, \top = \bot$ (for $y \neq \top$).
- if $a_1 > a_2$, then $(a_1, b_1) \,/ (a_2, b_2) = (a_1 - a_2, b_1 - b_2)$;
- if $a_1 = a_2$ and $b_1 \geq b_2$, then $(a_1, b_1) \,/ (a_2, b_2) = (0, b_1 - b_2)$;
- otherwise $(a_1, b_1) \,/ (a_2, b_2) = \bot$.

Case 1: a Formula Separating ACT and ACT_ω

Theorem

The formula $(p \wedge q \wedge (p \setminus q) \wedge (p \,/\, q))^+ \preceq p$ *(where* $A^+ = A \cdot A^*$*) is derivable in* $\mathbf{ACT}_\omega$*, but not in* $\mathbf{ACT}$*.*

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- Derivability in $\mathbf{ACT}_\omega$ is established straightforwardly, by the following "induction-in-the-middle" rule:

$$\frac{A \preceq B \quad A \cdot A \preceq B \quad A \cdot B \cdot A \preceq B}{A^+ \preceq B}$$

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- In order to prove non-derivability of this formula in $\mathbf{ACT}$, we construct an algebraic countermodel.

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- The model includes ∞ as the greatest element, and two subclasses of languages:
  - finite language over $\{a\}$;
  - languages of the form

$$A \cup \bigcup_{h \geq 0} \{a^i c a^{i+h} \mid i \geq f_R(h)\} \cup \bigcup_{h > 0} \{a^{i+h} c a^i \mid i \geq f_L(h)\},$$

    where A is a cofinite language over $\{a\}$ and $f_R, f_L : \mathbb{N} \to \mathbb{N}$ are functions of at least linear growth.

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      where A is a cofinite language over $\{a\}$ and $f_R, f_L : \mathbb{N} \to \mathbb{N}$ are functions of at least linear growth.
- Let p be the language of this form with $A = \{a\}^*$ and $f_R(h) = f_L(h) = 2h; \qquad q = p \cdot \{a\}$.

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- Thus we get $\{a\}^+ = \infty \not\preceq p$.
- The very general idea is a follows: by multiplying p on $\{a\}$ **on one side**, we shall shift the h argument and eventually violate the growth conditions.
- In the "induction-in-the-middle" scheme, however, h just oscillates ± 1.

Case 2: The Decomposition Rule without Join

- By *decomposition of the Kleene star* we mean the following inference rule:

$$\frac{\mathbf{1} \preceq \beta \quad \alpha \cdot \alpha^* \preceq \beta}{\alpha^* \preceq \beta}$$

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- Decomposition is essential for proving Σ_1^0-hardness of **ACT** (K. 2019).
- The decomposition rule follows from $\alpha^* = \mathbf{1} \vee \alpha \cdot \alpha^*$, which is derivable in **ACT**.
- But what if we do not have $\vee$ and $\wedge$?

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The decomposition rule

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- *Admissibility* of this rule in the *equational theory* is still an open question.
- We construct a model, namely, a residuated monoid with iteration (RMI), with two elements a and b, such that $\mathbf{1} \preceq b$, $a \cdot a^* \preceq b$, but $a^* \not\preceq b$.

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is not generally valid in residuated monoids with iteration.

- *Admissibility* of this rule in the *equational theory* is still an open question.
- We construct a model, namely, a residuated monoid with iteration (RMI), with two elements a and b, such that $\mathbf{1} \preceq b$, $a \cdot a^* \preceq b$, but $a^* \not\preceq b$.
- Notice that in the presence of the ω-rule decomposition is derivable, so our Kleene star should be non-standard.

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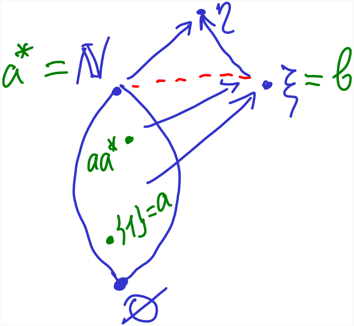
- η is the new maximum, and ξ is greater than any $x \in \mathscr{P}(\mathbb{N})$, except for $\mathbb{N}$ itself, with which it is incomparable.

Countermodel for Decomposition



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for $x, y \in \mathscr{P}(\mathbb{N})$

$\varnothing \cdot y = \varnothing$

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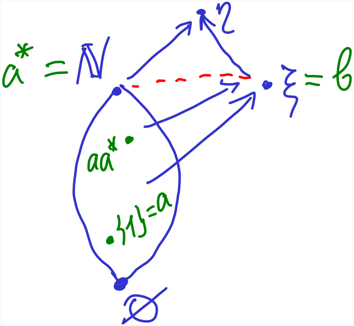

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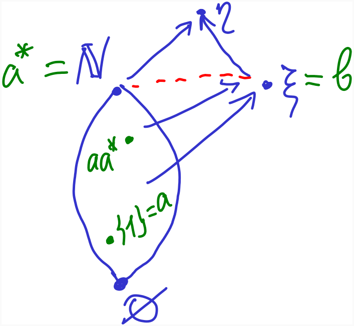

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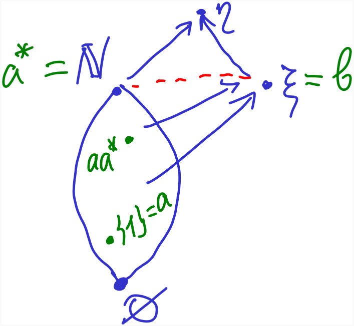

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Kleene star is defined in $\mathscr{P}(\mathbb{N})$ as usual; $\xi^* = \eta^* = \eta$.

For $a = \{1\}$ and $b = \xi$ we have $\mathbf{1} = \{0\} \preceq b$, $a \cdot a^* \preceq b$, but $a^* \npreceq b$.

Thanks*