Residually finite and equationally Noetherian groups

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In this talk

1. Definitions and examples
   - Residually finite (RF) groups
   - Equationally Noetherian (EN) groups
   - RF $\Leftrightarrow$ EN?
2. A few specific classes
3. The counterexamples
   - RF $\not\Rightarrow$ EN
   - EN $\not\Rightarrow$ RF

Residually finite groups

Definition

A group G is *residually finite* (*RF*) if for any $1 \neq g \in G$ there exists a homomorphism $\varphi\colon G \to F$, with F finite, such that $\varphi(g) \neq 1$. (Equivalently, $\bigcap\{H \leq G \mid [G:H] < \infty\} = \{1\}$.)

Properties and examples

- Finitely generated abelian groups are RF.
- (Malcev'40): finitely generated linear groups (subgroups of $GL_n(K)$ for a field K) are RF.
- Any finitely generated RF group G is *Hopfian*, i.e. any epimorphism $\varphi\colon G \to G$ is an isomorphism. Indeed, if not then $\bigcap\{H \leq G \mid [G:H] < \infty\} \supseteq \ker(\varphi) \supsetneq \{1\}$.
- Subgroups, finite extensions, direct and free products of RF groups are RF.

Equationally Noetherian groups

Let G be a group and $n \geq 1$. Any element $s \in F_n(X_1, \ldots, X_n) * G$ defines a *word map* $s\colon G^n \to G$; we calculate $s(g_1, \ldots, g_n)$ by replacing each X_i with g_i in s and evaluating in G.

Definition

For a *system of equations* $S \subseteq F_n * G$, we define its *solution set*,

$$V_G(S) := \{(g_1, \ldots, g_n) \in G^n \mid s(g_1, \ldots, g_n) = 1_G \text{ for all } s \in S\}.$$

We call a group G *strongly equationally Noetherian* (*strongly EN*) if for any $n \geq 1$ and any $S \subseteq F_n * G$, there exists a finite subset $S_0 \subseteq S$ such that $V_G(S_0) = V_G(S)$.

For finitely generated groups, EN $\Leftrightarrow$ strongly EN.

Equationally Noetherian groups

Properties and examples

- Abelian groups are strongly EN.
- (Baumslag–Myasnikov–Remeslennikov'99): linear groups (subgroups of $GL_n(R)$ for a commutative ring R) are EN.
- (Groves–Hull'17): any finitely generated EN group is Hopfian.
- Subgroups and direct products of EN groups are EN.
- (Baumslag–Myasnikov–Roman'kov'97): finite extensions of EN groups are EN.
- (Sela'10): free products of EN groups are EN.

Warnings:

- For any non-abelian EN group G, the groups $\bigoplus_{j=1}^{\infty} G$ and $\prod_{j=1}^{\infty} G$ are EN but not strongly EN.
- There is a (non-Noetherian) commutative ring R and a 2-step nilpotent subgroup of $GL_3(R)$ that is not strongly EN.

RF $\Leftrightarrow$ EN?

There are groups that are RF but not EN, or vice versa.

Examples

- Any abelian group is strongly EN, but many are not RF. For instance, $(\mathbb{Q}, +)$ is strongly EN but not RF.
- Any direct sum of finite groups is RF, but many such won't be EN. For instance, $\bigoplus_{n=1}^{\infty} S_n$ is RF but not EN.

What about finitely generated examples?

Theorem (Wilson'80)

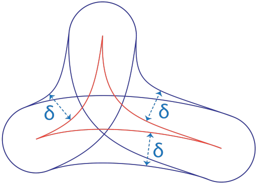
Any countable RF group G embeds in a 2-generated RF group $\widehat{G}$, which can be chosen so that if G is soluble, residually soluble or residually nilpotent, then so is $\widehat{G}$.

For instance, there is a 2-generated group (containing $\bigoplus_{n=1}^{\infty} S_n$) that is RF but not EN.

What about explicit examples?

Hyperbolic groups

We say a geodesic metric space X is δ-*hyperbolic* if triangles in X are 'δ-slim', and a finitely generated group is *hyperbolic* if it acts properly discontinuously and cocompactly on a δ-hyperbolic space.



Examples

- Finitely generated free groups: X = a tree.
- Surface groups, $\pi_1(\Sigma_g)$: $X = \mathbb{H}^2$.
- Small cancellation groups.
- 'Random' groups (Gromov'87).

Lots of nice properties: biautomaticity, solvable conjugacy problem, linear isoperimetric inequality, finite asymptotic dimension, . . .

Question (Gromov'87)

Are hyperbolic groups RF?

Theorem (Sela, Reinfeldt–Weidmann)

Hyperbolic groups are EN.

Relatively hyperbolic groups

We can generalise hyperbolic groups to *relatively hyperbolic groups*: groups G which become hyperbolic after 'coning off' a collection $\{H_1, \ldots, H_m\}$ of subgroups.

Examples

- $G = \pi_1(M)$, for M a non-compact hyperbolic 3-manifold of finite volume, is often hyperbolic relative to $H \cong \mathbb{Z}^2$.
- $G = H_1 * \cdots * H_m$.
- A group G acting cocompactly on a tree with finite edge stabilisers, with $\{H_j\} =$ (representatives of) vertex stabilisers.

Many properties of the H_j pass to G. Now if G is RF (or EN) then so are the subgroups H_j—but what about the converse?

Theorem (Osin'07)

Hyperbolic groups are RF $\iff$ if each H_j is RF, so is G.

Theorem (Groves–Hull'19)

If each H_j is EN, so is G.

Baumslag–Solitar groups

Given $p, q \in \mathbb{Z} \setminus \{0\}$, we define

$$BS(p,q) = \langle a, t \mid t^{-1} a^p t = a^q \rangle.$$

These come in four types:

- If $|p| = |q| = 1$, then $BS(p,q)$ is virtually abelian.
- If $|p| = 1$ or $|q| = 1$, then $BS(p,q)$ is metabelian.
- If $|p| = |q|$, then $BS(p,q)$ is virtually $F_k \times \mathbb{Z}$.
- Otherwise, $G = BS(p,q)$ can be non-Hopfian (e.g. $BS(2,3)$), Hopfian but not RF (e.g. $BS(2,4)$), ...

Theorem (Meskin'72, Baumslag–Myasnikov–Remeslennikov'99)

$BS(p,q)$ is RF $\iff \#\{1, |p|, |q|\} \leq 2 \iff BS(p,q)$ is EN.

Extensions of abelian groups

A group G is *polycyclic* (respectively *nilpotent*) if it has a subnormal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

with each quotient G_{i+1}/G_i cyclic (respectively central in G/G_i).

For $\mathcal{P} \in \{\text{nilpotent}, \text{polycyclic}\}$, we may consider an *abelian-by-$\mathcal{P}$ group*—that is, a group G with a normal abelian subgroup $N \trianglelefteq G$ such that G/N is $\mathcal{P}$.

Theorem (Hall'59, Jategaonkar'74)

Finitely generated abelian-by-~~nilpotent~~polycyclic groups are RF.

Theorem (Bryant'77)

Finitely generated abelian-by-nilpotent groups are EN.

Question (Bryant'77): what about abelian-by-polycyclic groups?

Abelian-by-polycyclic groups

Theorem (V'20)

Abelian-by-polycyclic groups are EN.

Idea of proof.

Suppose $A \trianglelefteq G$ is abelian and $P := G/A$ is polycyclic. Then G embeds in $A^P \rtimes P$; since A^P is abelian, it is thus enough to consider groups of the form $G = A \rtimes P$. Then A is a right $\mathbb{Z}[P]$-module.

Now if $s \in F_n$, $(a_1, \ldots, a_n) \in A^n$ and $(p_1, \ldots, p_n) \in P^n$, then

$$s(a_1 p_1, \ldots, a_n p_n) = 1_G \Leftrightarrow s(p_1, \ldots, p_n) = 1_P \wedge \sum_{j=1}^{n} a_j \cdot s_j(p_1, \ldots, p_n) = 0$$

where $s_1, \ldots, s_n \in \mathbb{Z}[F_n]$ depend only on s. There exists a map $\varepsilon \colon F_n \to Q$, with Q polycyclic, such that every map $F_n \to P$ factors through ε (Gonçalves–Wong'02), and so $s_j(p_1, \ldots, p_n) = \Gamma_{p_1, \ldots, p_n}(\hat{s}_j)$ for some $\Gamma_{p_1, \ldots, p_n} \colon \mathbb{Z}[Q] \to \mathbb{Z}[P]$ and $\hat{s}_j \in \mathbb{Z}[Q]$.

Let $S \subseteq F_n$. As P is EN, $V_P(S) = V_P(S_0)$ for a finite $S_0 \subseteq S$. Also, as $\mathbb{Z}[Q]$ is a Noetherian ring (Hall'54), the submodule of $\mathbb{Z}[Q]^n$ generated by $f(s) = (\hat{s}_1, \ldots, \hat{s}_n)$ for $s \in S$ is also generated by a finite subset $\{f(s) \mid s \in S_1\}$. Thus $V_G(S) = V_G(S_0 \cup S_1)$. □

Rigid soluble groups

Recall that a group G is *soluble* if it has a subnormal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$ with each G_{i+1}/G_i abelian.

Definition

A soluble group G, with a *normal* series as above, is *rigid* if each G_{i+1}/G_i is torsion-free as a $\mathbb{Z}[G/G_i]$-module.

For instance, free soluble groups are rigid.

Theorem (Romanovskii'09)

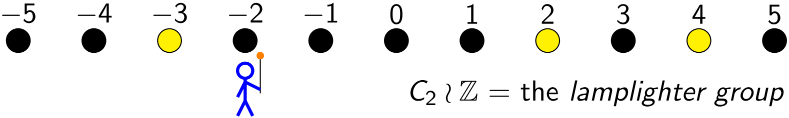
Rigid soluble groups are EN.

However, a *finitely generated* rigid soluble group embeds in an iterated wreath product $\mathbb{Z}^m \wr (\mathbb{Z}^m \wr (\cdots \wr (\mathbb{Z}^m \wr \mathbb{Z}^m) \cdots))$ (Myasnikov–Romanovskii'10), and so it is also RF.

Wreath products

Definition

Let G and H be groups. The (*restricted*) *wreath product* of G by H is the semidirect product $G \wr H := \left(\bigoplus_{h\in H} G\right) \rtimes H$, where the action of H is given by permutations of coordinates.



$C_2 \wr \mathbb{Z} =$ the *lamplighter group*

Theorem (Gruenberg'57)

The wreath product $G \wr H$ is RF if and only if both

- G and H are RF; and
- either G is abelian or H is finite.

Wreath products (continued)

What about being EN? If G and H are EN groups, then:

- if H is finite, then $G \wr H$ is a finite extension of the EN group G^H, and so is EN;
- if G is non-abelian and H is infinite, then $G \wr H$ is not strongly EN (Baumslag–Myasnikov–Roman'kov'97);
- if G is abelian and H is infinite, then the situation is more complicated...

Lemma (V'20)

If $G \neq 1$ and H has an infinite locally finite subgroup, then $G \wr H$ is not strongly EN.

For instance, if $G = C_2$ and $H = C_2 \wr \mathbb{Z}$ then $G \wr H = C_2 \wr (C_2 \wr \mathbb{Z})$ is RF but not EN.

Better than RF?

Definition

Let G be a group and $A \subseteq G$. We say A is *separable* in G if for all $g \in G \setminus A$ there exists $\varphi\colon G \to F$ with F finite and $\varphi(g) \notin \varphi(A)$.

- G is: *RF* if $\{1\}$ is separable;
- *conjugacy separable* if conjugacy classes are separable;
- *cyclic subgroup separable* if cyclic subgroups are separable.

Theorem (Remeslennikov'71)

$G \wr H$ is conjugacy separable if and only if it is RF and

- G and H are conjugacy separable;
- H is cyclic subgroup separable.

Theorem (V'20)

$G \wr H$ is cyclic subgroup separable if and only if it is RF and

- G and H are cyclic subgroup separable.

Thus, $C_2 \wr (C_2 \wr \mathbb{Z})$ is conjugacy- and cyclic subgroup separable.

Symplectic groups

Given a connected Lie group G, consider its universal cover $p\colon \widetilde{G} \to G$. We can lift the group structure on G to a group structure on $\widetilde{G}$. The subgroup $\ker(p) \trianglelefteq \widetilde{G}$ is then a discrete central subgroup isomorphic to $\pi_1(G)$.

Now consider $G = Sp_{2n}(\mathbb{R}) = \{A \in SL_{2n}(\mathbb{R}) \mid A^T \Omega A = \Omega\}$, where $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $p\colon \widetilde{G} \to G$ as above, and $H = p^{-1}(Sp_{2n}(\mathbb{Z}))$.
(Deligne'78) showed that H is not RF for $n \geq 2$. In particular, $Z = \ker(p|_H)$ can be identified with $\pi_1(Sp_{2n}(\mathbb{R})) \cong \mathbb{Z}$, and the subgroup $2\mathbb{Z}$ of Z belongs to all finite-index subgroups of H.

We can show that $Sp_{2n}(\mathbb{Z})$ is finitely presented, hence so is H.

Symplectic groups (continued)

Theorem (V'20)

Let G be a connected Zariski-closed Lie subgroup of $GL_n(\mathbb{R})$, and let $p\colon \widetilde{G} \to G$ be a covering homomorphism with $\widetilde{G}$ connected. Then $\widetilde{G}$ is strongly EN.

Idea of proof.

Let $S \subseteq F_n * \widetilde{G}$ be a system of *exponent sum zero* equations, i.e. such that X_i and X_i^{-1} appear the same number of times in each $s \in S$.

Let $V := V_G(p_*(S))$. Then each $s \in S$ defines a map $s_*\colon V \to \ker(p)$ (as s has exponent sum zero and $\ker(p)$ is central), and in particular $V_{\widetilde{G}}(S)$ is determined by its image under $p^{(n)}\colon \widetilde{G}^n \to G^n$.

V is an algebraic variety in some $\mathbb{R}^N$, and so V has finitely many path components: $V_1, \dots, V_r$, say (Delfs–Knebusch'81). As $\ker(p)$ is discrete, for each $s \in S$ the maps $s_*|_{V_j}$ are constant. Thus $p^{(n)}(V_{\widetilde{G}}(S)) = \bigcup_{j \in \mathcal{I}} V_j$.

But G is strongly EN, so $V = V_G(p_*(S_0))$ for a finite $S_0 \subseteq S$. For each j, either $V_j \subseteq p^{(n)}(V_{\widetilde{G}}(S))$, or there exists $s_j' \in S$ with $(s_j')_*|_{V_j} \equiv g_j \neq 1_{\widetilde{G}}$. It follows that $V_{\widetilde{G}}(S) = V_{\widetilde{G}}(S_0 \cap \{s_j'\})$. □

Thank you!