Automorphisms of the category of free algebras

Elena Aladova

Federal University of Rio Grande do Norte, Brazil

March 4, 2021

Introduction

Why we study automorphisms of some arbitrary algebraic structure?

- Automorphism group is interesting in itself.
- Applications: automorphism group is a group of symmetries.
- ...
- Universal Algebraic Geometry.

Classical Algebraic Geometry

systems of polynomial equations	$\Leftrightarrow$	*solutions*
$\begin{cases} x_1 + x_2 - x_3 = 0, \\ x_1^2 - x_2^3 = 0. \end{cases}$		$\{ (a_1, a_2, a_3) \mid a_i \in \mathbb{C} \}$
$f_k(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots x_n]$		$\mathbb{C}^n$
radical ideals		*algebraic sets*
the category of coordinate $\mathbb{C}$*-algebras (f.g. reduced* $\mathbb{C}$*-algebras)*		*the category of algebraic sets*

Universal Algebraic Geometry

Let Θ be a variety of algebras

systems of equations	$\Leftrightarrow$	***solutions***
$\begin{cases} x_1 x_2^3 = 1, \\ x_1^2 x_3^2 = x_2 x_4. \end{cases}$		$\{\ (a_1, a_2, a_3, a_4) \mid a_i \in H\ \}$
$f(x_1, \dots, x_n) \in F[x_1, \dots x_n]$		H^n
H-closed congruences		*algebraic sets*
the category of coordinate algebras		*the category of algebraic sets*

Universal Algebraic Geometry

Question

When do two algebras have the same algebraic geometry?

Let

- Θ be an arbitrary variety of algebras,
- H_1, H_2 be algebras from Θ,
- $Var(H_1) = Var(H_2) = \Theta$.

Definition

Algebras H_1 and H_2 are

geometrically similar
(have the same algebraic geometry)

if the categories of algebraic sets over H_1 and H_2 are isomorphic.

Universal Algebraic Geometry

An answer to the question above can be formulated in terms of two notions:

geometric equivalence of algebras

geometrically automorphical equivalence of algebras.

Geometrical Equivalence

Definition

Algebras are ***geometrically equivalent*** *if they have equal possibilities with respect to solving systems of equations.*

$$\left\{\begin{array}{l} f_1(x_1,x_2)=0,\\ f_2(x_1,x_2)=0,\\ \dots\\ f_k(x_1,x_2)=0, \end{array}\right. \Rightarrow \begin{array}{c} \{\ (a_1,a_2)\mid a_i\in H_1\ \}\\ \\ \{\ (b_1,b_2)\mid b_i\in H_2\ \} \end{array} \Rightarrow \begin{array}{c} \left\{\begin{array}{l} f_1'(x_1,x_2)=0,\\ f_2'(x_1,x_2)=0,\\ \dots \end{array}\right. \\ \\ \left\{\begin{array}{l} f_1''(x_1,x_2)=0,\\ f_2''(x_1,x_2)=0,\\ \dots \end{array}\right. \end{array}$$

Geometrical Equivalence

Proposition (B. Plotkin, 2003)

*Algebras H_1 and H_2 are **geometrically equivalent***

if and only if

each infinitary quasi-identity of the algebra H_1 is a quasi-identity of the algebra H_2.

$$\left(\bigwedge_i (w_i = w_i')\right) \longrightarrow (w_0 = w')$$

Geometrically Automorphical Equivalence

The notion of the geometrically automorphical equivalence of algebras is concerned with

automorphisms of the category Θ^0 of finitely generated algebras in Θ.

Speaking informally, algebras are

geometrically automorphical equivalent
if
they have equal possibilities with respect to solving systems of equations after some "transformation" of one of the given algebra.

This "transformation" is given by an automorphism of the category Θ^0.

Geometrical Equivalence and Geometrically Automorphical Equivalence

Theorem (B. Plotkin, 1997)

*Algebras are **geometrically equivalent***

if and only if

*they are **automorphically equivalent** for some **inner automorphism** of the category Θ^0 of free finitely generated algebras from Θ.*

Method of Verbal Operations

B. Plotkin, G. Zhitomirskii (2006), A. Tsurkov (2007 –).

The method of verbal operations provides a machinery to study the automorphisms group $Aut(\Theta^0)$.

It turns out that for the wide class of varieties an automorphism of the category Θ^0 can be represented as a decomposition of

an inner and a strongly stable automorphisms.

Moreover, there is one-to-one correspondence between

strongly stable automorphisms
and
special systems of words (elements of $F(X)$).

Condition on the variety Θ

We will assume that the following condition holds in our variety Θ:

Condition

Every automorphisms of Θ^0 takes a monogenic free algebra to an isomorphic one:

$$\Phi(F(x)) \cong F(x),$$

where $\Phi \in Aut(\Theta^0)$.

Definition

A variety Θ is said to have an "Invariant Basis Number", if for any natural numbers n, m an isomorphism between free algebras with n and m free generators implies that $n = m$.

$$F(X_1) \cong F(X_2) \Rightarrow |X_1| = |X_2|.$$

Condition on the variety Θ

Condition

Every automorphisms of Θ^0 takes a monogenic free algebra to an isomorphic one:

$$\Phi(F(x)) \cong F(x),$$

where $\Phi \in Aut(\Theta^0)$.

Proposition (Y. Katsov, R. Lipyanski, B. Plotkin, 2007)

Let Θ be an IBN-variety, Φ be an automorphism of the category Θ^0. Then $\Phi(F(X)) \cong F(X)$, $F(X) \in \Theta^0$.

Automorphism group $Aut(\Theta^0)$

The next theorem describes the structure of the automorphism group $Aut(\Theta^0)$.

Theorem (B. Plotkin, G. Zhitomirski, 2006)

The group $Aut(\Theta^0)$ can be decomposed as

$$Aut(\Theta^0) = Inn(\Theta^0)St(\Theta^0).$$

Remark

Some of the strongly stable automorphisms can be inner.

Inner Automorphisms of Θ^0

Definition

*An automorphism Φ of the category Θ^0 is **inner**, if*

1. *for every object $F(X)$ of Θ^0,*
$$\Phi(F(X)) \cong F(X);$$
2. *for every homomorphism $\nu : F(X_1) \to F(X_2)$, there exist isomorphisms $\beta^{\Phi}_{F(X_i)} : F(X_i) \to \Phi(F(X_i))$, ($i = 1, 2$), such that*
$$\Phi(\nu) = \beta^{\Phi}_{F(X_2)} \circ \nu \circ \left(\beta^{\Phi}_{F(X_1)}\right)^{-1}.$$

In Category Theory terms: *an automorphism Φ of a category Θ^0 is **inner**, if it is isomorphic as a functor to the identity automorphism of the category Θ^0.*

Strongly Stable Automorphisms of Θ^0

Let an automorphism Φ of Θ^0 and an object $F(X) \in \Theta^0$ be given.
For an element f in $F(X)$, there is unique homomorphism

$$\alpha_f : F(x_0) \to F(X)$$

such that

$$\alpha_f(x_0) = f.$$

Let $\delta^\Phi : F(x_0) \to \Phi(F(x_0))$ be an isomorphism. With the automorphism Φ we associate a bijection

$$s_F^{\delta^\Phi} : F(X) \to \Phi(F(X)) \tag{1}$$

defined as follows:

$$s_F^{\delta^\Phi}(a) = \Phi(\alpha_a) \circ \delta^\Phi(x_0). \tag{2}$$

Strongly Stable Automorphisms of Θ^0

Definition

*An automorphism Φ of the category Θ^0 is called **strongly stable** if it satisfies the conditions:*

1. *for every object $F(X)$ of Θ^0,*

$$\Phi(F(X)) = F(X);$$

2. *for every homomorphism $\nu : F(X_1) \to F(X_2)$, there exist bijections $s^{\delta^\Phi}_{F(X_i)}$, (i=1,2), such that*
   2.1 $\Phi(\nu) = s^{\delta^\Phi}_{F(X_2)} \circ \nu \circ \left(s^{\delta^\Phi}_{F(X_1)}\right)^{-1}$,
   2.2 $s^{\delta^\Phi}_{F(X_i)} \mid_{X_i} = id_{X_i}$.

Method of Verbal Operations

It provides a machinery to calculate the group $St(\Theta^0)$.

There is one-to-one correspondence between strongly stable automorphisms and special systems of words.

strongly stable automorphisms	$\Leftrightarrow$	*special systems of words*

Method of Verbal Operations

Let

- $F(x_1, \ldots, x_m)$ be a free algebra in Θ,
- $v = v(x_1, \ldots, x_m) \in F(x_1, \ldots, x_m)$,
- H be an arbitrary algebra from Θ.

Definition

The operation $\omega_v : H^m \to H$ determined as

$$\omega_v(h_1, \ldots, h_m) = v(h_1, \ldots, h_m)$$

*is called a **verbal operation** on the algebra H defined by the word v.*

Method of Verbal Operations

- Ω is the signature of the variety Θ.
For example, if $\Theta = Grp$, then $\Omega = \{1, ^{-1}, \cdot\}$
- For every symbol $\omega \in \Omega$ of arity $k > 0$ we can pick up a word $v \in F(x_1, \ldots, x_k)$ and define on the set $F(x_1, \ldots, x_n)$ a verbal operation ω_v:

$$\omega_v(f_1, \ldots, f_k) = v(f_1, \ldots, f_k), \tag{3}$$

where $f_1, \ldots, f_k \in F(x_1, \ldots, x_n)$.
- The "word" for a nullary operation ω_0 is taken from the subalgebra generated by ω_0.
If $\Theta = Grp$ and $\Omega = \{1, ^{-1}, \cdot\}$, then $1^* = 1$.
If $\Theta = Com - K$ and $\Omega = \{0, 1, -, \cdot_\alpha, +, \cdot\}$, then $0^* = 0$, $1^* \in K$.

Method of Verbal Operations

For every symbol of arity $k > 0$ $\omega_i \in \Omega$	$\Rightarrow$	we pick up a word in $F(x_1, \ldots, x_k)$ v_i	$\Rightarrow$	define on $F(X)$ the operation ω_{v_i}

$$\Downarrow$$

System of words
$W = \{v_1, v_2, \ldots\}$

Original operations and Verbal operations

$$\Omega = \{\omega_1, \omega_2, \ldots\} \quad \Rightarrow \quad \left\{ \Omega_W = \{\omega_{v_1}, \omega_{v_2}, \ldots\} \right\}_W$$

Examples of Verbal Operations

Let $\Theta = Grp$ and $\Omega = \{1, ^{-1}, \cdot\}$.

$\omega = \cdot$	$v \in F(x,y)$	$\omega_v = \bullet$
	$y \cdot x$	$x \bullet y = y \cdot x$
	$x \cdot y \cdot x$	$x \bullet y = x \cdot y \cdot x$
	$x \cdot x$	$x \bullet y = x \cdot x$

$$\Omega = \{1, ^{-1}, \cdot\} \;\Rightarrow\; W_1 = \{1,\ x^{-1},\ y \cdot x\} \;\Rightarrow$$

$$\begin{cases} 1^{*_1} & = & 1 \\ x^{\bullet_1} & = & x^{-1} \\ x \bullet_1 y & = & y \cdot x \end{cases} \;\Rightarrow\; \Omega_{W_1} = \{1, ^{\bullet_1}, \bullet_1\},$$

$$\Omega = \{1, ^{-1}, \cdot\} \;\Rightarrow\; W_2 = \{1,\ x^{5},\ y \cdot x \cdot y\} \;\Rightarrow$$

$$\begin{cases} 1^{*_2} & = & 1 \\ x^{\bullet_2} & = & x^{5} \\ x \bullet_2 y & = & y \cdot x \cdot y \end{cases} \;\Rightarrow\; \Omega_{W_2} = \{1, ^{\bullet_2}, \bullet_2\},$$

Method of Verbal Operations

Original operations and Verbal operations

$$\Omega = \{\omega_1, \omega_2, \dots\} \quad \Rightarrow \quad \left\{ \Omega_W = \{\omega_{v_1}, \omega_{v_2}, \dots\} \right\}_W$$

$$F(X) = \big(F(X), \omega_1, \omega_2, \dots\big) \qquad F_W(X) = \big(F(X), \omega_{v_1}, \dots\big)$$

Method of Verbal Operations

Definition

A system of words $W = \{v_\omega \mid \omega \in \Omega\}$ *is called* ***strongly stable*** *if for every* $F(X) \in \Theta^0$ *there exists an isomorphism*

$$\varphi_{F(X)} : F(X) \to F_W(X)$$

such that

$$\varphi_{F(X)} \mid_X = id_X.$$

Theorem (B. Plotkin, G. Zhitomirskii, 2006; A. Tsurkov, 2007)

There is one-to-one correspondence between strongly stable systems of words and strongly stable automorphisms.

Variety of Semigroups

- ✓ The signature $\Omega = \{\cdot\}$.
- ✓ $F(x_1, \ldots, x_n)$ is a free semigroups.

For the symbol of arity $k = 2$ $\{\cdot\}$	$\Rightarrow$	we pick up a word $v_\cdot \in F(x_1, x_2)$	$\Rightarrow$	define on $F(x_1, \ldots, x_n)$ a verbal operation $\bullet$

$\Downarrow$

$$W = \{v_\cdot\} = \{x_1^{i_1^{(1)}} x_2^{i_2^{(1)}} x_1^{i_1^{(2)}} x_2^{i_2^{(2)}} \cdots x_1^{i_1^{(n)}} x_2^{i_2^{(n)}}\},$$

where $i_1^{(1)}, i_2^{(1)}, i_1^{(2)}, i_2^{(2)}, \ldots, i_1^{(n)}, i_2^{(n)} \in \mathbb{N} \cup \{0\}$.

Variety of Semigroups

- ✓ $F(X) = \big(F(X),\ \cdot\big)$ is a free semigroup.
- ✓ $F_W(X) = \big(F(X),\ \bullet\big)$ is an (universal) algebra.

$$x_1 \bullet x_2 = x_1^{i_1^{(1)}} x_2^{i_2^{(1)}} x_1^{i_1^{(2)}} x_2^{i_2^{(2)}} \cdots x_1^{i_1^{(n)}} x_2^{i_2^{(n)}}$$

Definition

A system of words $W = \{v_\omega \mid \omega \in \Omega\}$ is ***strongly stable*** *if for every $F(X) \in \Theta^0$ there exists an isomorphism*

$$\varphi_{F(X)} : F(X) \to F_W(X)$$

such that

$$\varphi_{F(X)} \mid_X = id_X .$$

Variety of Semigroups

$$\begin{cases} W_1 = \{x_1 \cdot x_2\} & \Rightarrow \quad \Phi_1 \text{ is inner} \\ W_2 = \{x_2 \cdot x_1\} & \Rightarrow \quad \Phi_2 \text{ is not inner,} \\ & \qquad \text{(G. Mashevitzky, B. Plotkin, 2007)} \end{cases}$$

$$St(\Theta^0) \cong \mathbb{Z}_2.$$