Hardness of equations over finite solvable groups under the exponential time hypothesis

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[1] Joint work with Paweł Idziak, Piotr Kawałek, and Jacek Krzaczkowski.

Equations in groups

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$$X + X = 1$$

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W. l. o. g. of the form

$$\alpha = 1$$

for an expression $\alpha \in (G \cup \mathcal{X} \cup \mathcal{X}^{-1})^*$ (with variables $\mathcal{X}$).

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In many infinite groups these problems are undecidable!

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- if yes, then α is not an identity.

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- $\mathrm{EQN\text{-}SAT}(G/H) \leq_T^{\mathsf{P}} \mathrm{EQN\text{-}SAT}(G)$
- if H is a verbal subgroup, then $\mathrm{EQN\text{-}SAT}(H) \leq_m^{\mathsf{P}} \mathrm{EQN\text{-}SAT}(G)$
- But: there are monoids $N \leq M$ such that $\mathrm{EQN\text{-}SAT}(M) \in \mathsf{P}$ but $\mathrm{EQN\text{-}SAT}(N)$ is NP-complete

Related Problems

TermID(G)

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Roughly: like EQN-SAT but variables are restricted to two values.

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$\textsc{TermID}(G) \leq_m^{\mathsf{P}} \text{EQN-ID}(G) \leq_T^{\mathsf{P}} \text{EQN-SAT}(G) \leq_m^{\mathsf{P}} \textsc{ProgramSAT}(G)$

Systems of equations

Theorem (Goldmann, Russell, 2002)

- *If G is non-abelian, satisfiability of systems of equations in G is* NP *complete.*
- *If G is abelian, satisfiability of systems of equations in G is in* P.

Overview: complexity of equations in finite groups

Theorem (Goldmann, Russell, 2002)

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- *If G is nilpotent, then* $\mathrm{EQN\text{-}SAT}(G) \in \mathsf{P}$.
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If G is non-solvable, then $\mathrm{EQN\text{-}ID}(G)$ *is* coNP*-complete.*

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solvable, non-nilpotent	in NP	in coNP
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Theorem (Földvári, Horváth 2020)

- $\mathrm{EQN\text{-}SAT}(Q \rtimes A) \in \mathsf{P}$ *for Q a p-group, A abelian.*

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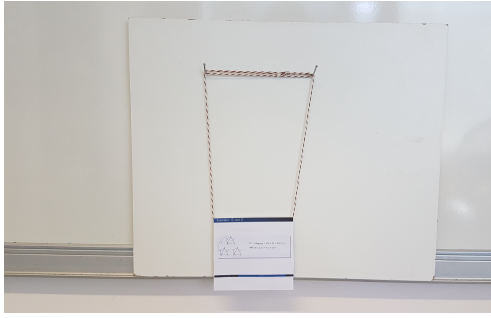

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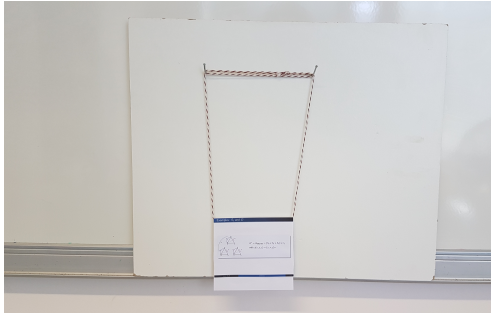

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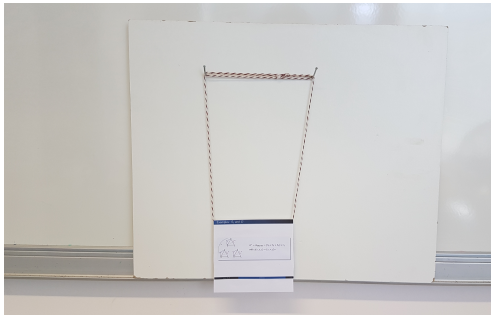
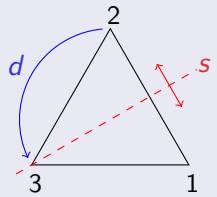


Commutators: $[x, y] = x^{-1}y^{-1}xy = \begin{cases} ?? & \text{if } x \neq 1 \text{ and } y \neq 1 \\ 1 & \text{otherwise.} \end{cases}$

Examples: S_3 and G^*



$S_3 =$ group of permutations over three elements

$=$ symmetry group of a regular triangle

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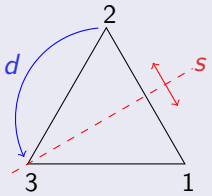

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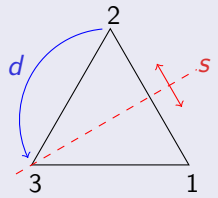

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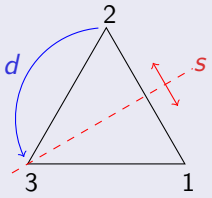

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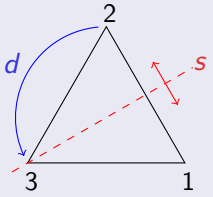

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X^{-1}Y^{-1}XY = (1\,2\,3) \qquad X^{-1}Y^{-1}XY = (1\,2) \qquad Y(1\,2\,3)XXY = (1\,3\,2)
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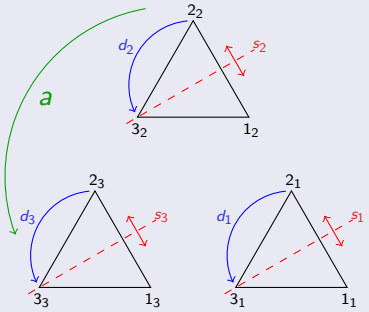

$$G^* = G_{648,705} = S_3 \wr C_3 = (S_3 \times S_3 \times S_3) \rtimes C_3$$

with $a(x, y, z) = (z, x, y)a$

The Fitting length

Commutators: $[x, y] = x^{-1}y^{-1}xy$ and $[x_1, \ldots, x_k] = \big[[x_1, \ldots, x_{k-1}], x_k\big]$

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The Fitting length $\mathrm{FitLen}(G)$ (nilpotent length) of G is the smallest k such that there are normal subgroups

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G$$

with N_i/N_{i-1} nilpotent for all $i = 1, \ldots, k$.

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$\mathrm{FitLen}(S_3) = 2$: $1 \lhd C_3 \lhd S_3$ with $S_3/C_3 = C_2$

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Main results

Theorem (W., ICALP 2020)

Let G be finite solvable group and assume that either

- $\mathrm{FitLen}(G) \geq 4$, *or*
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(S_4 = symmetric group on 4 elements)

Main results

Theorem (Idziak, Kawałek, Krzaczkowski, W.)

Let G be finite solvable group of Fitting length $d \geq 3$. Then $\mathrm{EQN\text{-}SAT}(G)$ *and* $\mathrm{EQN\text{-}ID}(G)$ *cannot be decided in time* $2^{o(\log^{d-1} N)}$ *under ETH.*

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C-Coloring

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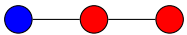

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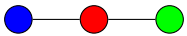

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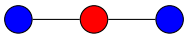

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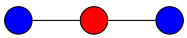

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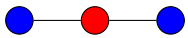

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- NP-complete for $C \geq 3$
- 3-Coloring cannot be solved in time $2^{o(|V|+|E|)}$ unless ETH fails
  (see e. g. Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, Saurabh, Thm. 14.6).
- $\rightsquigarrow$ for every $C \geq 3$, C-Coloring cannot be solved in time $2^{o(|V|+|E|)}$ unless ETH fails.

Reduce 2-COLORING to EQN-SAT(S_3)

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Length: $|\beta| \approx 2^m$.

$$[d, \alpha_1] = d^{-1}{\alpha_1}^{-1} d \alpha_1$$
$$[d, \alpha_1, \alpha_2] = {\alpha_1}^{-1} d^{-1} \alpha_1 d \alpha_2^{-1} d^{-1} {\alpha_1}^{-1} d \alpha_1 \alpha_2$$
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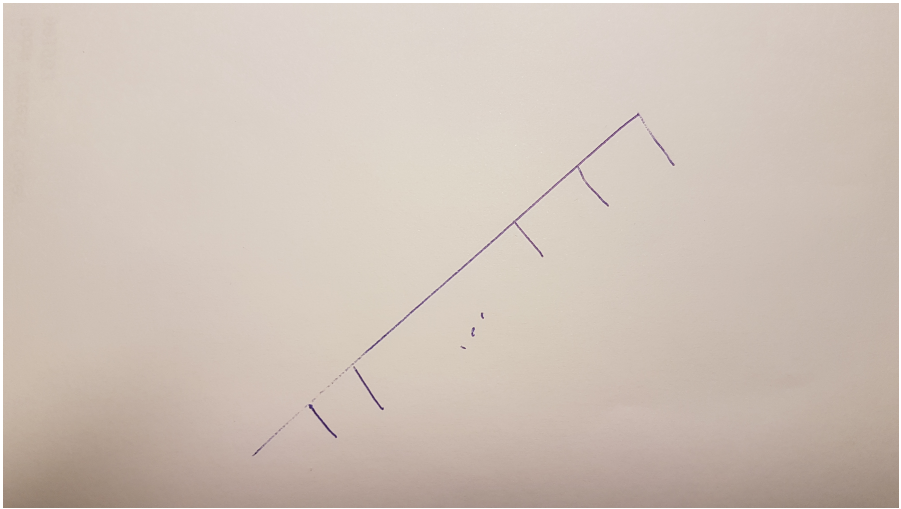
Reduce the size of the equation



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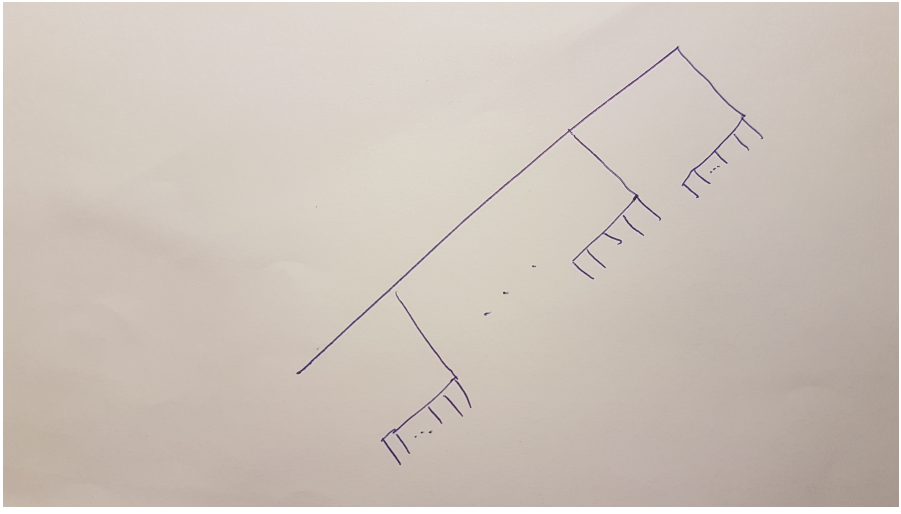

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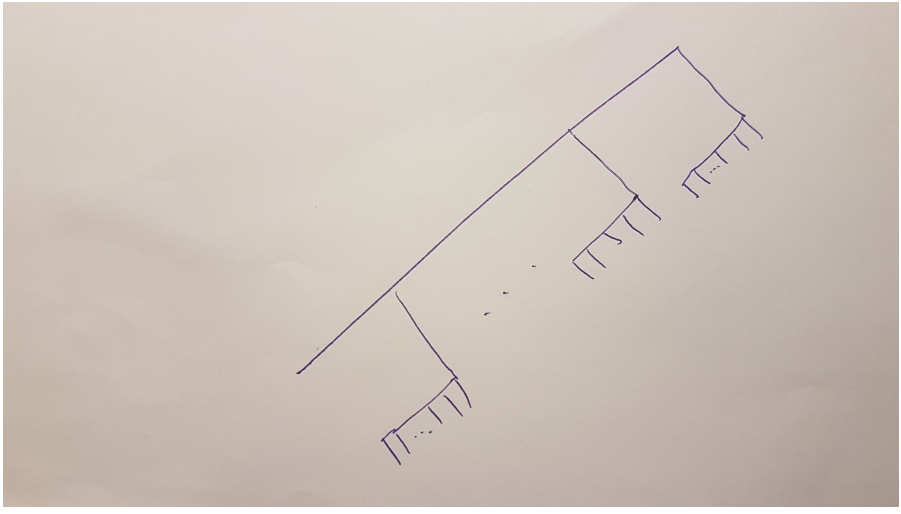

$\rightsquigarrow$ need Fitting length 3

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Key Observation

$|\beta_k| \approx 2^\mu \rightsquigarrow |\gamma| \approx 2^\mu \cdot 2^\mu \approx 2^{2\sqrt{m}}$

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Reduce 3-Coloring to EQN-SAT(G^*)

Recall: $G^* = (S_3 \times S_3 \times S_3) \rtimes C_3$

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Two expressions as input.

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If a semi-group S has a group divisor of Fitting length at least 3, then $\mathrm{EQN\text{-}SAT}(S)$ is not in P *under ETH.*

G is a divisor of S if G is a quotient of a sub-semigroup of S.

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Theorem (Almeida Volkov, Goldberg, 2009)

If G is a maximal subgroup of S, then $\textsc{TermID}(G) \leq_m^{\mathsf{P}} \textsc{TermID}(S)$.

G-programs

$\textsc{ProgramSAT}(G)$

Constant: The group G
Input: a G-program $P \in (\mathcal{X} \times G \times G)^*$
Question: $\exists$ an assignment $\sigma : \mathcal{X} \to \{0,1\}$ s.t. $\sigma(P) = 1$?

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If the n-input AND function can be computed via G-programs of polynomial length, then $\textsc{ProgramSAT}(G \wr C_k)$ is NP-complete (for $k \geq 4$).

Does a similar result hold for EQN-SAT or EQN-ID?

Conclusion / Open Problems

- Quasipolynomial lower bound for $\mathrm{EQN\text{-}SAT}(G)$ and $\mathrm{EQN\text{-}ID}(G)$ under ETH if G if of Fitting length 3.
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Thank you!