# Generic complexity of the identity problem over finite groups and monoids

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**The identity problem over group** $G$ is the following:

- Given a term $t$.
- Is $t = 1$ an identity over $G$?

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**Identity**: $xyx = xyxyx$.

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**Volkov (1985)** found finite bases for more general classes of Brandt semigroups.

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**Sapir (1994)**

Let $S$ be a fixed finite group (semigroup). What is the computational complexity of the identity problem over $S$?

# Feasible problems

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- Monoids with less than 6 elements (Klima).

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- 6-element Brand monoid $B_2^1$ (Klima, Seif).
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- Some 0-simple semigroups (Plescheva, Vertesi).

# Generic-case approach

Kapovich, Myasnikov, Schupp and Shpilrain in 2003 developed generic approach to algorithmic problems, which considers an algorithmic problem on "most"of the inputs (i.e., on a generic set) instead of the entire domain and ignores it on the rest of inputs (a negligible set). It turned out, that many famous undecidable problems are easily decidable on most inputs.

# Generic sets

Let $I$ be the set of all inputs and $I_n$ be the set of all inputs of size $n$ (sphere of radius $n$). For a subset $S \subseteq I$ define the following sequence

$$\rho_n(S) = \frac{|S \cap I_n|}{|I_n|}, \ n = 1, 2, 3, \ldots$$

**The asymptotic density** of set $S$ is the following limit (if it exists)

$$\rho(S) = \lim_{n \to \infty} \rho_n(S).$$

$S$ is called **generic** if $\rho(S) = 1$ and **negligible** if $\rho(S) = 0$.

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We will deal with words and pairs of words over a finite alphabets. Let $A$ be a finite alphabet. Denote by $A^*$ the set of all finite words in alphabet $A$. For every word $w \in A^*$ we will denote by $|w|$ the length of word $w$. The size of pair $(w_1, w_2) \in A^* \times A^*$ is $|w_1| + |w_2|$.

# Generic algorithms

Algorithm $\mathcal{A} : I \to I \cup \{?\}$ is called **generic** if

1. $\mathcal{A}$ halts on all inputs from $I$,
2. set $\{x \in I : \mathcal{A}(x) = ?\}$ is negligible.

# Generic algorithm for groups

**Theorem**

Let $G$ be a finite group. Then the identity problem over $G$ is generically decidable in polynomial time.

# Generic algorithm for groups

One can define for every term $t \in \{X \cup X^{-1}\}^*$ and for every $i = 1, \ldots, n$, where $n = |t|$, $d_i(t)$ as sum of powers of letter $x_i$ in word $t$.

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3. If YES, then output the answer "?".
4. If NO, then output the answer "NO".

# Generic algorithm for monoids

**Theorem**

Let $S$ be a finite monoid such that there is an element $a \in S$ of period greater than 1. Here period of $a$ is a minimal natural $m$ such that $a^k = a^{k+m}$ for some $k$. Then the identity problem over $S$ is generically decidable in polynomial time.

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One can define for every pair of terms $p, q \in X^*$ and for every $i = 1, \ldots, n$, where $n = |p| + |q|$, $d_i(p, q)$ as the number of letter $x_i$ in word $p$ minus the number of letter $x_i$ in word $q$.

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3. If YES, then output the answer "?".
4. If NO, then output the answer "NO".

# Gromyko type algorithms

Andrei Andreyevich Gromyko – Soviet Minister of Foreign Affairs, known as «Mister No».

# Brandt semigroups

Let $G$ be a group, $I$ a set with at least 2 elements, and $0 \notin G \cup I$. Define a multiplication operation on the set $B(G, I) = I \times G \times I \cup \{0\}$ as follows:

$$(i, g, j)(k, h, l) = \begin{cases} (i, gh, l), & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

for all $i, j, k, l \in I$ and all $g, h \in G$. Also $0x = 0$ and $x0 = 0$ for all $x \in B(G, I)$. The set $B(G, I)$ with defined multiplication is called the **Brandt semigroup over the group** $G$ with **index set** $I$.

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We are interested in finite Brandt semigroup $B_n = B(E, \{1, \ldots, n\})$ over the trivial one-element group $E$, which elements are just pairs $(i, j)$ of natural numbers.
Monoid $B_n^1$ is semigroup $B_n$ with jointed unit 1.

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**Theorem**

The problem of identity checking over the Brandt monoid $B_n^1$ is decidable generically in polynomial time.

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**Theorem**

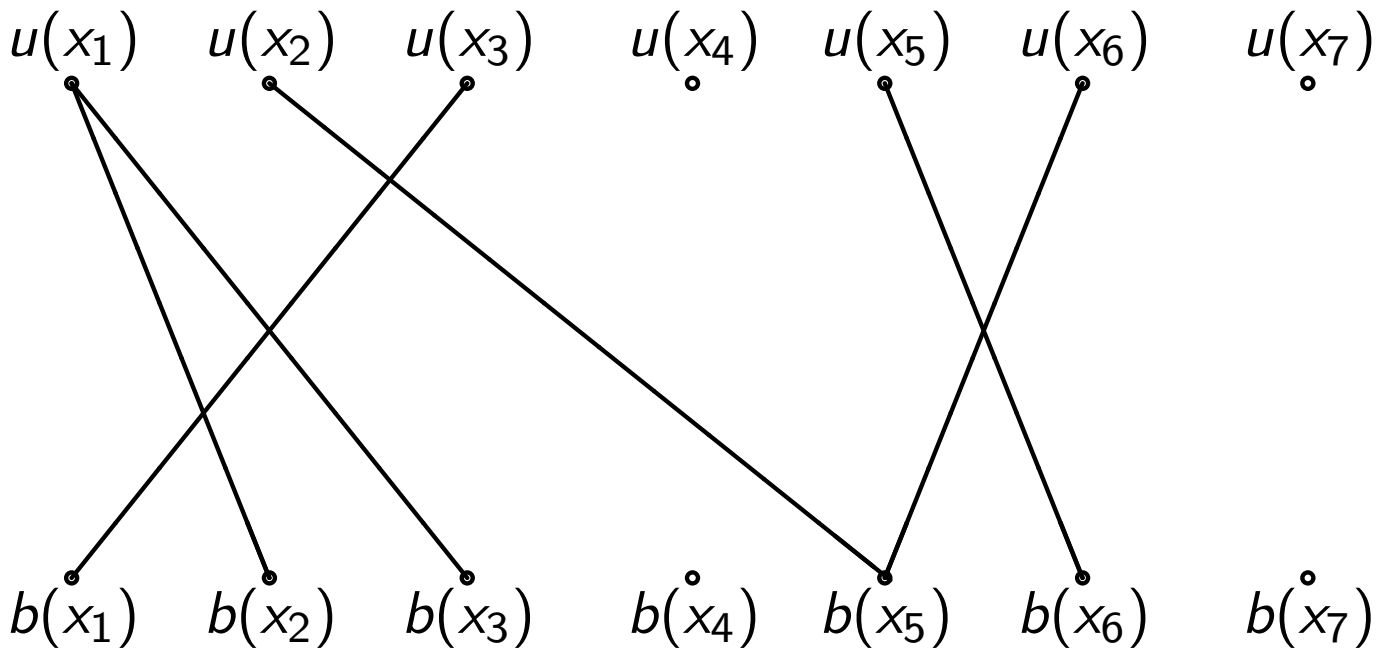
The problem of identity checking over Plescheva-Vertesi semigroup is decidable generically in polynomial time.

# Graphs

For a term $t(x_1, \ldots, x_n)$ define the following bipartite graph $B(t)$. It has $n$ upper vertices $u(x_1), \ldots, u(x_n)$ and $n$ bottom vertices $b(x_1), \ldots, b(x_n)$. Each variable $x_i$ corresponds to two vertices $u(x_i)$ and $b(x_i)$. Vertices $u(x_i)$ and $b(x_j)$ are connected by an edge in $B(t)$ if and only if there is a subword $x_i x_j$ in the term $t$.

# Example of graph

For example the following picture is the graph $B(t)$ of term $t = x_1x_3x_1x_2x_5x_6x_5$.



# Connected components

Denote by

$$C(B(t)) = \{C_1, C_2, \ldots, C_k\}$$

the set of all connected components (given by sets of vertices) of the graph $B(t)$.

# The identity problem over $B_n$

Seif and Szabo proved that $t_1 = t_2$ is identity over Brandt semigroup $B_n$ if and only if

1. $C(B(t_1)) = C(B(t_2))$,
2. connected component of $B(t_1)$, containing $u(x_i)$, where $x_i$ is the first letter of $t_1$, is the same as connected component of $B(t_2)$, containing $u(x_j)$, where $x_j$ is the first letter of $t_2$,
3. connected component of $B(t_1)$, containing $b(x_k)$, where $x_k$ is the last letter of $t_1$, is the same as connected component of $B(t_2)$, containing $b(x_l)$, where $x_l$ is the last letter of $t_2$.

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Note that sets $C(B(t_1))$ and $C(B(t_2))$ can be constructed in polynomial time, therefore the problem of identity checking in Brandt semigroup $B_n$ is polynomial time decidable. But, as proved by Klima and Seif, the problem of identity checking over Brandt monoid $B_n^1$, $n \geq 2$, is co-NP-complete. Thus there is no effective polynomial algorithm for the problem of identity checking over Brandt monoid $B_n^1$, $n \geq 2$, provided P $\neq$ NP.

# Scetch of the proof

**Lemma**

The set

$$\mathcal{EPT} = \{(t_1, t_2) \in \mathcal{PT} \ : \ C(B(t_1)) = C(B(t_2))\}.$$

is negligible.

# Scetch of the proof

**Lemma**

The set

$$\mathcal{EPT} = \{(t_1, t_2) \in \mathcal{PT} \ : \ C(B(t_1)) = C(B(t_2))\}.$$

is negligible.

1. Construct $B(t_1)$ and $B(t_2)$.
2. If $B(t_1) = B(t_2)$ then output "YES".
3. If $C(B(t_1)) \neq C(B(t_2))$ then output "NO".
4. If $C(B(t_1)) = C(B(t_2))$ then output "?".

# The end

Thank you!