Chains of centralizers and equationally noetherian groups

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Equations in groups

$\mathcal{L}_{gr} = \{\cdot^{(2)}, ^{-1}, 1\}$ be the standard group language

Let G be a group

We will consider equations in the language $\mathcal{L} = \mathcal{L}_{gr} \cup \{G\}$, also known as Diophantine group equations.

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ – be a finite set of variables, $t(\mathbf{x})$ is a term of the language $\mathcal{L}$ in variables $\mathbf{x}$.

Examples of terms are: $xy^{-2}g$, $[x, a]$, $x_1 \ldots x_n$.

Equations in groups

A formula of the type $t(\mathbf{x}) = 1$ is called an equation of unknowns $\mathbf{x}$. Any set of equations of the language $\mathcal{L}$ in unknowns $\mathbf{x}$ is called a system of equations in a group G.

Examples. $[x, y] = 1,\ x^n = 1,\ [x, g] = 1$.

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Two systems of equations $S_1(\mathbf{x})$ and $S_2(\mathbf{x})$ of the language L are called equivalent over G if their solution sets coincide.

Noetherness by equations

Definition. *An algebraic structure A is called equationally noetherian, if for any natural n any system of equations $S(\boldsymbol{x})$ in n variables $\boldsymbol{x}$ is equivalent to its finite subsystem $S_0(\boldsymbol{x}) \subset S(\boldsymbol{x})$.*

E.Yu.Danyarova, A.G. Miasnikov, V.N. Remeslennikov Algebraic geometry over algebraic structures, SB RAS Publishing, Novosibirsk, 2016.

Examples

Equationally noetherian algebraic structures:

- Abelian groups;
- Finite groups;
- Hyperbolic groups;
- Finitely generated nilpotent groups

Equationally nonnoetherian algebraic structures:

- $A wr B$, A – nonabelian, B – infinite.
  G. Baumslag, A.Miasnikov, V.Roman'kov Two theorems about equationally noetherian groups, JoA, 1997
- $\prod G$, when G – nonabelian group.
  M. Shahryari, A.Shevlyakov Direct products, varieties, and compactness conditions, GCC, 2017;
- Some nilpotent groups, metabelian groups...
- Some monoids, semigroups...
- Some infinite graphs, orders, hypergraphs...

Kotov lemma

Kotov M.V. ON EQUATIONALLY NOETHERIAN PROPERTY, Herald of Omsk University, 2013, 2, 24-28.

Lemma. *An algebraic structure* $A = \langle A, \mathcal{L} \rangle$ *is not equationally noetherian iff, there exist series* $(a_i)_{i \in \mathbb{N}}$, $a_i \in A^n$, *and series of equations* $(s_i(x))_{i \in \mathbb{N}}$ *of the language* $\mathcal{L}$ *such that* $A \not\models s_i(a_i)$ *for any* i, *and* $A \models s_j(a_i)$ *for any* $j < i$.

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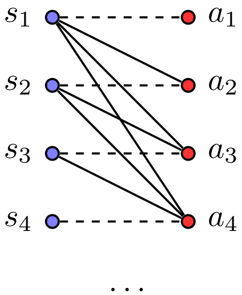

Examples. Simple graphs. Infinite clique

Let $L = \{E(x, y)\}$, $!E(x, x)$.

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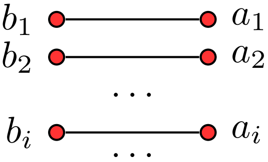
$$
\begin{cases}
E(x, a_1) & a_1 \\
E(x, a_2) & a_2 \\
\vdots & \vdots \\
E(x, a_i) & a_i \\
\vdots & \vdots
\end{cases}
$$

Therefore an infinite clique is not equationally noetherian.

Infinite B-graph



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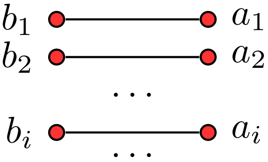

$$\begin{cases} E(x, b_1) & a_1 \\ E(x, b_2) & a_2 \\ \vdots & \vdots \\ E(x, b_i) & a_i \\ \vdots & \vdots \end{cases}$$

Infinite direct product of nonabelian groups

Let $G = \prod_i^{\infty} G_i$ and $a_i = (1, \ldots, 1, a_i', 1, \ldots) \in G$ where $a_i' \in G_i$ and a_i' is non-central element of G_i. Then G is not equationally noetherian.

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Definition. Let n be a natural number. We will call that algebraic structure A is n-equationally noetherian if any system of equations over n unknowns is equivalent to their finite subsystem.

G. Baumslag, A. Miasnikov, V. Remeslennikov, Algebraic geometry over groups I. Algebraic sets and ideal theory. Journal of Algebra, **219** *(1999) 16-79.*

Problem. *Let a group* $G = \langle G, L_G \rangle$ *is* 1*-equationally noetherian. Does it follow that* G *is equationally noetherian?*

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Description of all non-noetherian simple graphs (joint work with I.Buchinsky).

Centralizer dimension

Let $g \in G$, then the centralizer of g: $C(g) = \{h \in G | [g, h] = 1\}$

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Then there exist elements $m_1 \in M_1, \ldots m_i \in M_i, \ldots$ and
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$$\begin{cases} [x, m_1] & = & 1 & & a_0 \\ [x, m_2] & = & 1 & & a_1 \\ \vdots & & & & \vdots \\ [x, m_i] & = & 1 & & a_{i-1} \\ \vdots & & & & \vdots \end{cases}$$

Theorem 1. Let G be a group. If there exists strictly descending chain of centralizer of infinite length in G then:

1. G isn't equationally noetherian in one variable;
2. G isn't q_ω-compact.

Centralizer dimension to equations

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Consequence. *Let* $\{G_i | i \in I,\}$, *where* $|I| = \infty$ *and* G_i *is nonabelian group for any* i. *Then* $G = \prod_I G_i$ *isn't* q_ω*-compact group.*

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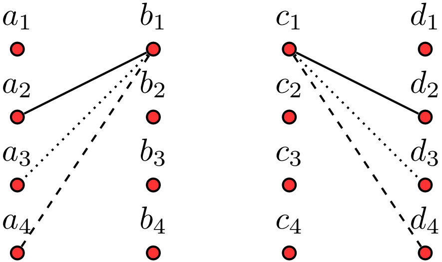
$$C(m_1) > C(m_1, m_2) > \ldots > C(m_1, \ldots, m_n)$$

Main example

$$G = \langle a_1, \ldots, b_1, \ldots, c_1, \ldots, d_1, \ldots \,|\, [a_i, b_j] = [d_i, c_j], i > j \rangle_{\mathfrak{N}(2,\mathbb{Z})}$$



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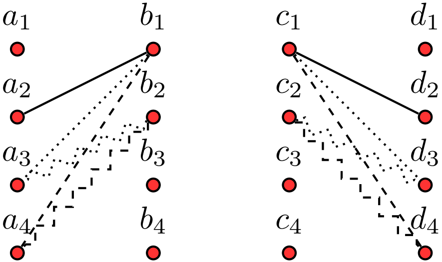

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Picture

The group G isn't equationally noetherian, since there exists a system S and series $(a_i, b_i) \in G^2$, such that Kotov's lemma is holds for S and (a_i, b_i):

$$\begin{cases} [x, b_1][y, c_1] & = & 1 & \qquad\qquad (a_1, d_1) \\ [x, b_2][y, c_2] & = & 1 & \qquad\qquad (a_2, d_2) \\ \vdots & & & \qquad\qquad \vdots \\ [x, b_i][y, c_i] & = & 1 & \qquad\qquad (a_i, d_i) \\ \vdots & & & \qquad\qquad \vdots \end{cases}$$

One variable equations

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But $Cdim(G) = 2$. How to check it?

We can do it directly by computing centralizers of elements of the group G. $C(x) = \langle x \rangle \cdot G'$ if $x \notin Z(G)$.

More interesting approach

A. Duncan, I. Kazachkov, V. Remeslenikov *Centralizer dimension and universal classes of groups*, // Siberian Electronic Mathematical Reports, 3, (2006), 197-215.

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Notice that the subgroup generated by $a_1, a_2, \ldots, b_1, b_2, \ldots$ is isomorphic to the subgroup generated by $c_1, \ldots, d_1, \ldots$ and both they are isomorphic to free infinitely generated two step nilpotent group F. Let $H_1 = \langle [a_i, b_j] | i > j \rangle$ and $H_2 = \langle [d_i, c_j] | i > j \rangle$.

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The groups H_1 and H_2 are isomorphic to each other.

Therefore G is a free nilpotent product of two infinite free groups F with amalgamation by the subgroup H_1 and H_2:

$$G = (F *_{H_1 = H_2} F)/[[x, y], z]$$

In this case the result follows from Theorem 3.5 and Proposition 3.6 from [DKR]

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Proposition *Let G and H be a torsion free two step nilpotent groups, which is equationally noetherian in one variable. Then $(G * H)_{N_2}$ is 1-equationally noetherian too.*

Further work

- Describe non-noetherian binary predicate structures
- The notion of equationally noetherian property for exact type of equations.
- Create checklist of equations for groups to check equationally noetherian property or prove that it doesn't exist
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Thanks for attention!