First order complexity of subgraph isomorphism

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First-order language

- FO sentences are built of
  relational symbols $\sim, =$;
  logical connectives $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$;
  variables $x, y, x_1, ...$;
  quantifiers $\forall, \exists$.

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- FO property $\mathcal{C}$ is define by a FO formula ϕ:
  $G \in \mathcal{C} \Leftrightarrow G \models \phi$.

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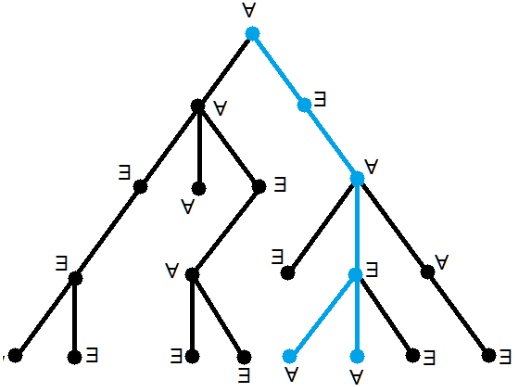

Quantifier depth and variable width

Variable width $W(\phi)$ —
number of different variables of ϕ

$$\phi = \exists x\,(\exists y\,[x \sim y] \wedge$$
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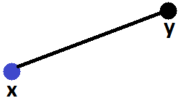

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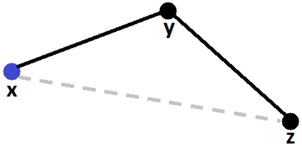

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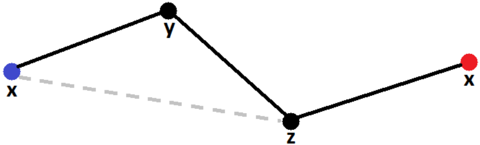

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Some examples

- property of a graph to be complete

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- property of a graph to have chromatic number k;
- property of a graph to be complete;
- property of a graph to have even number of vertices.

Subgraph Isomorphism

For a FO property $\mathcal{S}$, let

- $D(\mathcal{S})$ — minimal quantifier depth of a FO sentence defining $\mathcal{S}$,
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Remark: $W(\mathcal{S}) \leq D(\mathcal{S})$.

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F — fixed graph on ℓ vertices.

$\mathcal{S}(F)$ — set of **all** graphs containing an isomorphic copy of F.
Problem: find $D(F) = D(\mathcal{S}(F))$ and $W(F) := W(\mathcal{S}(F))$.
Trivial upper bound: $W(F) \le D(F) \le \ell$.

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Observation

No FO sent. ϕ with $W(\phi) < \ell$ distinguish between K_ℓ and $K_{\ell-1}$.
Therefore, $D(F) = W(F) = \ell$.

Induced Subgraph Isomorphism

$\mathcal{S}[F]$ — set of **all** graphs containing an **induced** isomorphic copy of F.
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Is it true that $W[F] = D[F] = \ell$ for every F? No!

Let $F = K_3 + e$ (*paw*).

Olariu, 1988

A graph G is paw-free if and only if each connected component of H is triangle-free or complete multipartite.

$D[F] \le 3$, because the following sentence defines $\mathcal{S}[F]$:

$$\exists x_1([\exists x_2 \exists x_3([x_1 \sim x_2] \wedge [x_1 \sim x_3] \wedge [x_2 \sim x_3])] \wedge$$
$$[\exists x_2([x_1 \nsim x_2] \wedge [\exists x_3([x_1 \sim x_3] \wedge [x_2 \sim x_3])] \wedge$$
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- SI and ISI problems for a pattern graph on ℓ vertices are solvable in time $O(n^\ell)$ on n-vertex input graphs by **exhaustive search**.

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- Courcelle, 1990: *every graph property definable by a sentence in monadic second-order logic can be efficiently decided on graphs of bounded treewidth.*

  In particular, for Subgraph Isomorphism, Courcelle's theorem implies time bound $f(\ell, tw) \cdot n$ for any class of input graphs having treewidth at most tw.

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connectivity

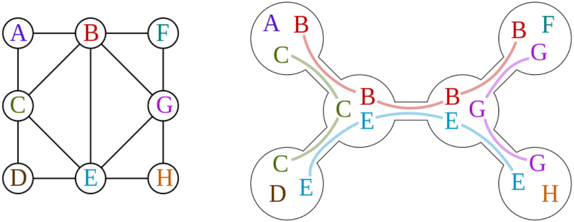
Let G be a graph with at least $k+1$ vertices.
G is k-connected if it is connected, and remains connected after removal of any $k-1$ vertices.
The connectivity $\kappa(G)$ is equal to the maximum k such that G is k-connected.

Treewidth

A tree decomposition of a graph $G = (V, E)$ is a tree T with vertices $X_1, \ldots, X_n$ such that

- $X_i \subset V$ for all i, and $X_1 \cup \ldots \cup X_n = V$;
- if $v \in X_i \cap X_j$, then all nodes X_k in the unique path path between X_i and X_j contain v;
- for every edge $v \sim w$ in E, there is i such that $v, w \in X_i$.

The width of T equals $\max_i |X_i| - 1$. The treewidth $tw(G)$ is the minimum width among all possible tree decompositions of G.



Relaxations for SI

Let π be a graph parameter.

- $D^k_\pi(F)$ — minimum quantifier depth of a FO sentence defining $\mathcal{S}(F)$ over connected graphs G with $\pi(G) \geq k$. For every k, $D^k_\pi(F) \geq D^{k+1}_\pi(F)$.
  Denote $D_\pi(F) = \min_k D^k_\pi(F)$ — *minimum quantifier depth of a FO sentence defining* $\mathcal{S}(F)$ *over connected graphs with sufficiently large values of* π.
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Inequalities $\kappa(G) \leq tw(G) < v(G)$ imply

$$D_v(F) \geq D_{tw}(F) \geq D_\kappa(F);$$

$$W_v(F) \geq W_{tw}(F) \geq W_\kappa(F).$$

Subgraph Isomorphism for connected input-graphs of large size: general results

Theorem

$D_v(F) \le \frac{2}{3}\ell + 1$ *for infinitely many connected* F.

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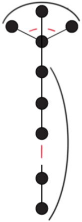

$S_{q,p}$

Theorem

$D_v(S_{q,p}) \leq \max\{p, \frac{1}{2}p + q - 2 - \frac{1}{2}p \mod 2\} + 2.$
$W_v(S_{q,p} \geq \max\{p, \frac{1}{2}p + q - 2 - \frac{1}{2}p \mod 2\}.$

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- If $S = S_{q,p}$, v is the end vertex of the tail part of S, $\deg v \neq 2$, then S is a pendant sparkler subgraph.

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Theorem

$W_v(F) \geq \ell - 1 - \min\{s(F), p(F), sp(F) - 2\}$.

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- If $\delta(F) \geq 2$, then $D_v(F) = W_v(F) = \ell$.

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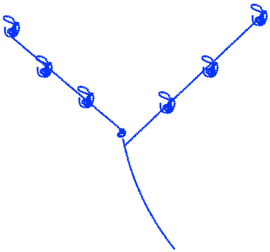
- If $\delta(F) \geq 2$, then $D_v(F) = W_v(F) = \ell$.

  In particular, for every $\ell \geq 3$,
  $D_v(C_\ell) = W_v(C_\ell) = D_v(K_\ell) = W_v(K_\ell) = \ell$.

Subgraph Isomorphism for input-graphs width large treewidth or connectivity: general results

Theorem

If F is connected, then $W_{tw}(F) \geq tw(F) + 1$ unless F is contained in some 3-megastar $M_{3,b}$.



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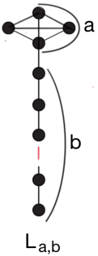
Theorem

If $e(F) > v(F)$, then

$$D_\kappa(F) \geq \frac{e(F)}{v(F)} + 2, \quad W_\kappa(F) \geq \frac{e(F)}{v(F)} + 1.$$

Subgraph Isomorphism for input-graphs with large treewidth or large connectivity: sequences



Theorem

For $a \geq 3 \quad D_{tw}(L_{a,b}) = W_{tw}(L_{a,b}) = D_{\kappa}(L_{a,b}) = W_{\kappa}(L_{a,b}) = a.$

In particular, $\quad D_{tw}(K_{\ell}) = W_{tw}(K_{\ell}) = D_{\kappa}(K_{\ell}) = W_{\kappa}(K_{\ell}) = \ell.$

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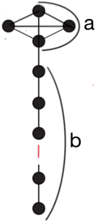

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In particular, $\quad D_{tw}(K_{\ell}) = W_{tw}(K_{\ell}) = D_{\kappa}(K_{\ell}) = W_{\kappa}(K_{\ell}) = \ell.$

The minimal degree of a k-connected graph is at least k. Therefore, if F is a tree, then $D_{\kappa}(F) = W_{\kappa}(F) = 1$.

Subgraph Isomorphism: small graphs

$\mathbf{F}$	$(\mathbf{W}_\kappa, \mathbf{D}_\kappa)$	$(\mathbf{W}_{tw}, \mathbf{D}_{tw})$	$(\mathbf{W_v}, \mathbf{D_v})$
$\mathbf{P_3}$	$(1,1)$	$(1,1)$	$(1,1)$
$\mathbf{K_3}$	$(3,3)$	$(3,3)$	$(3,3)$
$\mathbf{P_4}$	$(1,1)$	$(1,1)$	$(2,3)$
$\mathbf{K_{1,3}}$	$(1,1)$	$(1,1)$	$(3,4)$
$\mathbf{C_4}$	$(4,4)$	$(4,4)$	$(4,4)$
$\mathbf{L_{3,1}}$	$(3,3)$	$(3,3)$	$(3,3)$
$\mathbf{K_4 \setminus e}$	$(4,4)$	$(4,4)$	$(4,4)$
$\mathbf{K_4}$	$(4,4)$	$(4,4)$	$(4,4)$

Subgraph Isomorphism: arbitrary relations

- B. Rossman, 2016: $W_{\mathrm{Arb}}(F) \leq tw(F) + 3$, $D_{\mathrm{Arb}}(F) \leq td(F) + 2$.

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- O. Grigoryan, M. Makarov, MZ, 2020:
  For a tree F,

$$D_<(F) \le \tfrac{1}{2}\ell + \lceil \log_2(\ell + 2) \rceil - 1,\ W_<(F) \le \tfrac{1}{2}\ell + 2.$$

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  For a tree F,

  $$D_<(F) \le \frac{1}{2}\ell + \lceil \log_2(\ell + 2) \rceil - 1, \; W_<(F) \le \frac{1}{2}\ell + 2.$$

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Subgraph Isomorphism: arbitrary relations

- B. Rossman, 2016: $W_{\text{Arb}}(F) \le tw(F) + 3$, $D_{\text{Arb}}(F) \le td(F) + 2$.

  Tree-depth of G is the minimum height of a forest F with the property that every edge of G connects a pair of nodes that have an ancestor-descendant relationship to each other in F.

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Induced Subgraph Isomorphism: general result

Theorem

For every graph F on $\ell \geq 2$ vertices,

$$W[F] \geq \max \left\{ \left\lceil \frac{1}{2}\ell - 2\log_2 \ell \right\rceil + 2, \chi(F), \frac{e(F)}{v(F)} + 1 \right\}.$$

Induced Subgraph Isomorphism: 3 and 4 vertices

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Theorem

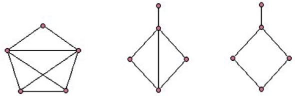
- *For every F on $\ell = 3$ vertices, $D[F] = W[F] = \ell$.*
- $D[F] = W[F] = \ell$ *for every F on $\ell = 4$ vertices unless $F = L_{3,1}$.*
  $W[L_{3,1}] = D[L_{3,1}] = 3$.

Induced Subgraph Isomorphism: 5 vertices

Theorem (E. Kudryavtsev, M. Makarov, A. Shlychkova, MZ; 2019)

For every F on $\ell = 5$ vertices, $D[F] \in \{\ell - 1, \ell\}$.

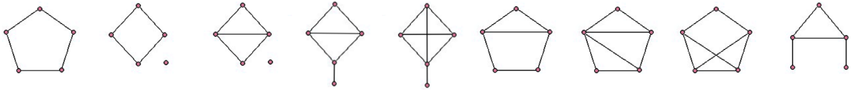
Graphs with $D[F] = 4$:



Graphs with $D[F] = 5$:

K_5 $K_{1,4}$ $K_{2,3}$ $K_{1,1,3}$ $K_{1,2,2}$ $K_{1,1,1,2}$

Remaining graphs:



Induced Subgraph Isomorphism: large graphs

Theorem (E. Kudryavtsev, M. Makarov, A. Shlychkova, MZ; 2019)

- *If F or $\overline{F}$ contains a connected component isomorphic to $L_{3,1}$, then $D[F] \leq \ell - 1$.*

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- *If F is complete multipartite or a disjoint union of isomorphic complete multipartite graphs, then $W[F] = \ell$.*

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$W[F] \geq \chi(F)$: the proof

• *k-th extension axiom E_k* — the sentence of quantifier depth k saying that, for every two disjoint sets $X, Y \subset V(G)$ with $|X \cup Y| < k$, there is a vertex $z \notin X \cup Y$ adjacent to all $x \in X$ and non-adjacent to all $y \in Y$.

Well-known fact

If G and H have k-extension property, then no FO sentence ϕ with $W(\phi) \leq k$ distinguish between between G and H.

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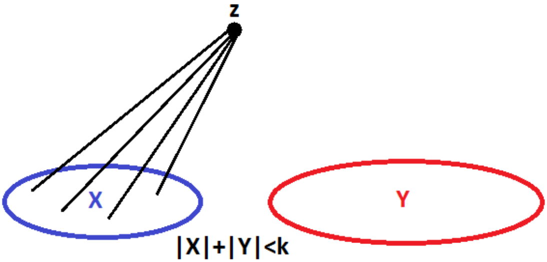

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Let $k = \chi(F) - 1$. *It is sufficient to construct graphs* G, H *such that* $G \sqsupset F$, $H \not\sqsupset F$ *and* $G \models E_k$, $H \models E_k$.

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Claim

Let $K_{n \times k}$ be the complete k-partite graph with each part consisting of n vertices. A.a.s. $H := G(K_{n \times k}, 1/2) \models E_k$.

$\chi(H) \leq \chi(K_{n \times k}) = k < \chi(F)$. So, $H \not\sqsupset F$. $\square$