Parameterized complexity of the word search problem in the Baumslag–Gersten group

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Joint work with A. Miasnikov

Plan:

– Parameterized complexity. Fixed parameter tractability.

– Word problem, word search problem in groups.

– Fixed parameter tractability of the word problem.

– Word search problem in Baumslag–Solitar group.

– Word search problem in Baumslag–Gersten group.

– Further developments.

Parameterized complexity

Downey, Fellows 1995: study complexity as a function of the size of the input, n, and a parameter of input or output, k.

Example 1. Given a graph on n vertices, is there a dominating set of size k?
(Set $D \subseteq V$ s.t. every vertex is in D or has a neighbor in D.)

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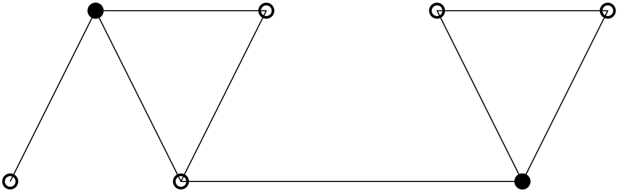

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Example 2. Given a graph on n vertices, is there a vertex cover of size k?
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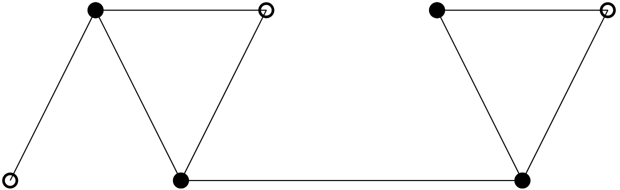

Can be solved in time $O(1.2738^k + kn)$.

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VERTEX COVER, solvable in time $O(1.2738^k + kn)$.
DOMINATING SET, solvable in time $O(n^{1+k})$.

Definition. A computational problem is fixed parameter tractable if there is an algorithm that solves the problem on input of size n with parameter k in time $f(k)n^c$, where $f : \mathbb{N} \to \mathbb{N}$ is computable and c is a constant.
Class of such problems is denoted FPT.

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Parameterized complexity allows:

– Finer classification of hard problems.

– Distinction between laborious and complicated problems.

Example. List all binary words of length n.

This is not a polynomial time solvable problem. If the size of answer $k = 2^n$ is taken as length of input, then the time is kn.

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Word problem

Word problem in a group G with presentation $\langle X \mid R \rangle$:

Given a group word $w \in (X \cup X^{-1})^*$, decide if $w = 1$ in G.

Equivalently: decide if there are $r_1, \ldots, r_k \in R \cup R^{-1}$ and $u_1, \ldots, u_k \in (X \cup X^{-1})^*$ s.t.

$$\begin{aligned} w &= (u_1^{-1} r_1 u_1) \cdot (u_2^{-1} r_2 u_2) \cdots (u_k^{-1} r_k u_k) = \\ &= r_1^{u_1} \cdot r_2^{u_2} \cdots r_k^{u_k} \end{aligned}$$

after free cancellation.

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Example. Consider abelian group $G = \langle a, b \mid a^{-1}b^{-1}ab \rangle$.
Input: $w = a^{-3}b^{-1}abab^{-2}ab^2$. In G, $w = 1$.
Indeed,

$$a^{-3}b^{-1}abab^{-2}ab^2 = a^{-2}(a^{-1}b^{-1}ab)a^2 \cdot (a^{-1}b^{-1}ab) \cdot b^{-1}(a^{-1}b^{-1}ab)b.$$

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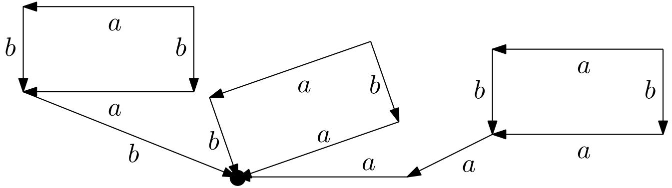

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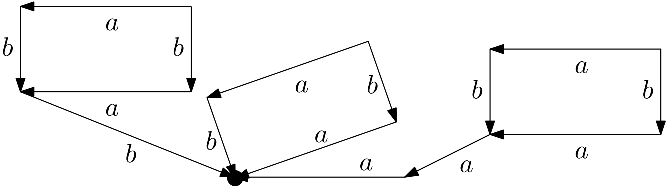
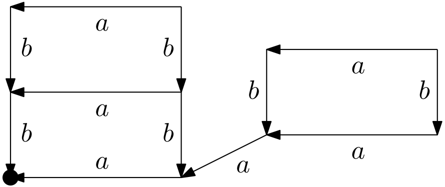
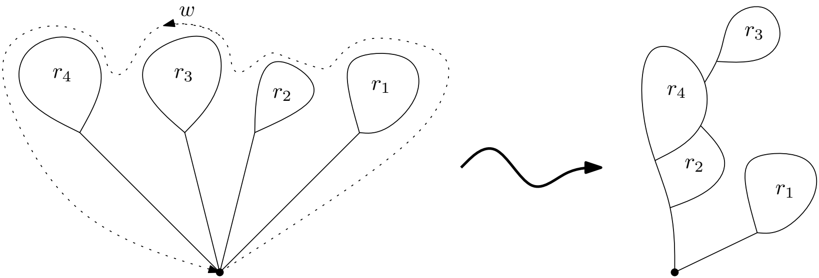




Word problem

Van Kampen diagrams (disc diagrams):



Homotopically: a disc.
Homeomorphically: a tree made of discs.

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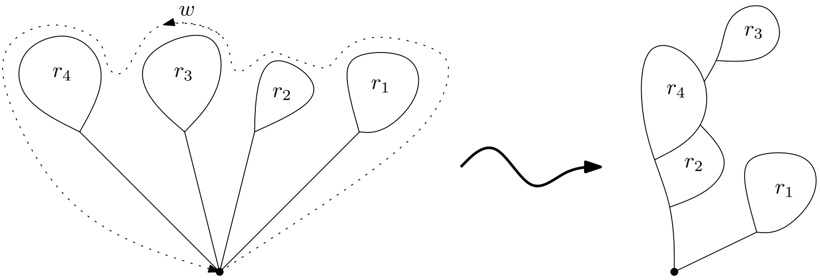

The minimal value k (area of the minimal van Kampen diagram) can serve as a parameter of the word (search) problem.
Length of the RHS in

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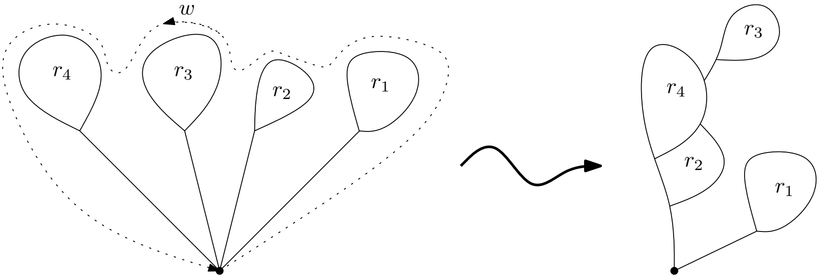

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Dehn function

Dehn function of the presentation $\langle X \mid R\rangle$:

$$f(n) = \max\{k \mid w = 1, |w| \leq n\}.$$

Tells how large an area is required to show that $w = 1$ if $|w| \leq n$.

Effect of parameterization

There are finite presentations where the word problem is:
– undecidable (Novikov 1955, Boone 1958),
– NP-complete (Sapir 2002).

Theorem 1. In every finite presentation the word problem can be solved in time $f(k)n^2$, where n is the size of input, k is the area of a van Kampen diagram, and f depends on the presentation.

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Proof. – Fix k. Van Kampen diagram is a tree of $\leq k$ subdiagrams *homeomorphic to a disc*, of area $\leq k$ each.

– There are only finitely many, $g(k)$, van Kampen diagrams of area $\leq k$ homeomorphic to a disc.

Algorithm: 1. Find a subword in w from the list of $g(k)$ boundary words of homeomorphic disc diagrams.

2. Remove the subword, perform free cancellation.

3. Repeat until whichever happens first: Step 1 fails, or the empty word is reached, or area runs out. □

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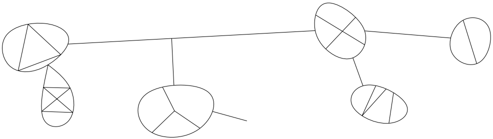

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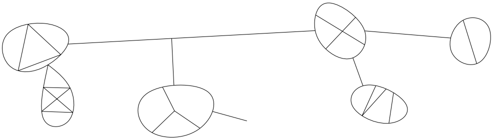

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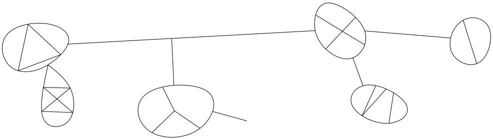

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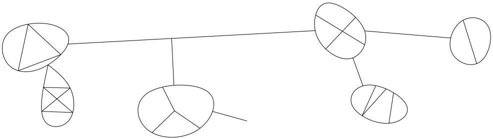

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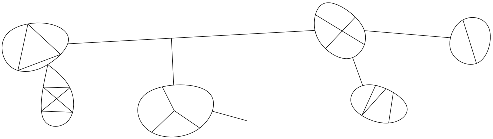

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Remark. $g(k)$, and therefore $f(k)$, is exponential even for $\langle a, b \mid a^{-1}b^{-1}ab \rangle$.

Therefore, the word problem and word search problem are FPT.

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Baumslag–Solitar groups

Baumslag–Solitar group $BS(m,n) = \langle a, b \mid (a^m)^b = a^n \rangle$.
If $|m| \neq |n|$, then Dehn function of $BS(m,n)$ is exponential. For example, in $BS(1,2) = \langle a, b \mid a^b = a^2 \rangle$ the equality

$$b^{-N}ab^N \cdot a = a \cdot b^{-N}ab^N, \quad N = 1, 2, \ldots.$$

requires $O(2^N)$ cells.

The word search problem in $BS(1,2) = \langle a, b \mid a^b = a^2 \rangle$ requires exponential time.

This contrasts the simplicity of the algorithm to solve the word (search) problem.

$BS(1,2)$ can be viewed as an HNN-extension of $\langle a \rangle$ with a stable letter b and associated subgroups $\langle a \rangle$, $\langle a^2 \rangle$.

The algorithm follows Britton's lemma for HNN-extensions:

1. Look for a subword $b^{-1}a^n b$, replace with a^{2n}; or a subword $ba^{2n}b^{-1}$, replace with a^n.
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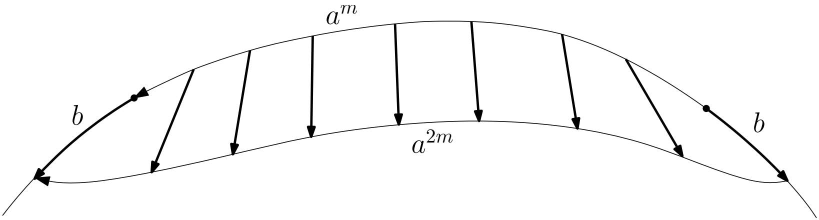
Theorem 2. Parameterized complexity of the word search problem in Baumslag–Solitar group $BS(1,2)$ is polynomial in n and k, where n is the size of input and k is the area of a van Kampen diagram.

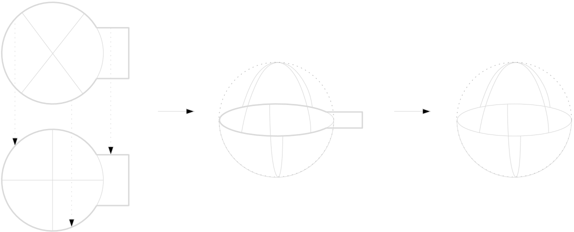
Proof. Each replacement $b^{-1}a^nb \to a^{2n}$ or $ba^{2n}b^{-1} \to a^n$ "peels off" a piece of van Kampen diagram.



Since area is k, this can happen $\leq k$ times.
If we count, we naively get time $O((n+k)^3)$. $\square$

Fine print: how do we know that we chew through the *minimal* van Kampen diagram? Generally, the same word may have different van Kampen diagrams.

If the same word has two substantially different van Kampen diagrams, we can glue them in a sphere:



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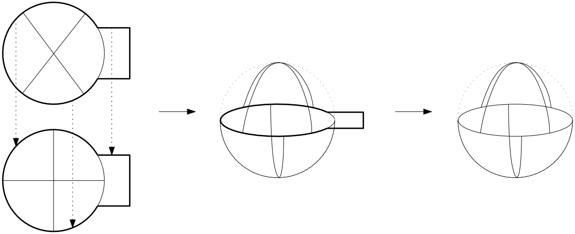

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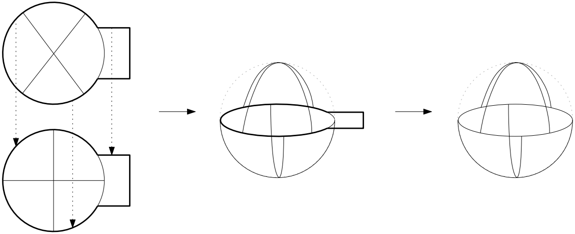

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$$\langle a, t \mid a^{a^t} = a^2 \rangle.$$

Rewrite as

$$GB = \langle a, b, t \mid a^b = a^2, a^t = b \rangle,$$

an HNN-extension of $BS(1,2) = \langle a, b \mid a^b = a^2 \rangle$ with a stable letter t and associated subgroups $\langle a \rangle$, $\langle b \rangle$.

Gersten 1992: GB has non-elementary Dehn function (in fact, $\underbrace{\exp \circ \exp \circ \cdots \circ \exp}_{\log n} 1$, Platonov 2004).

So the word search problem in this group is *non-elementary*.
However: word problem is polynomial (Myasnikov, Ushakov, Won 2011).

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$$\langle a, t \mid a^{a^t} = a^2 \rangle.$$

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$$GB = \langle a, b, t \mid a^b = a^2, a^t = b \rangle,$$

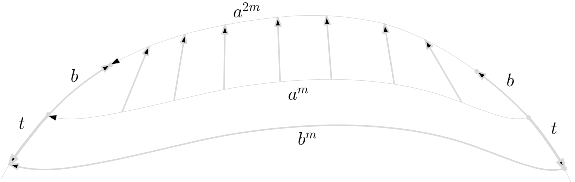
an HNN-extension of $BS(1,2) = \langle a, b \mid a^b = a^2 \rangle$ with a stable letter t and associated subgroups $\langle a \rangle$, $\langle b \rangle$.

Gersten 1992: GB has non-elementary Dehn function (in fact, $\underbrace{\exp \circ \exp \circ \cdots \circ \exp}_{\log n} 1$, Platonov 2004).

So the word search problem in this group is *non-elementary*.
However: word problem is polynomial (Myasnikov, Ushakov, Won 2011).

Theorem 3. Parameterized complexity of the word search problem in the Baumslag–Gersten group GB is polynomial in n and k, where n is the size of input and k is the area of a van Kampen diagram.

Proof. GB is aspherical and essentially the same algorithm works.



– We still apply Britton lemma repeatedly, looking for subwords $t^{-1} \dots t$ or $t \dots t^{-1}$.

– This time we have to find subwords that represent elements of $\langle a \rangle$ or $\langle b \rangle$ in $BS(1,2)$ (i.e., need to solve membership problem for those subgroups).

– Still polynomial time. □

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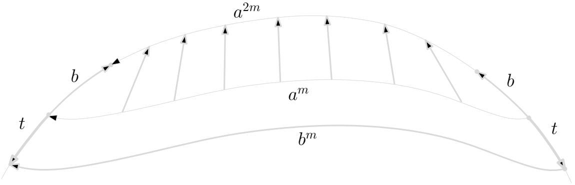

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The Baumslag–Gersten group

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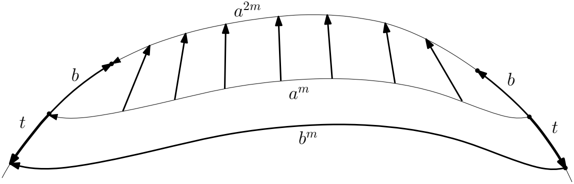

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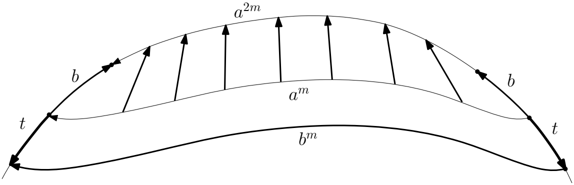

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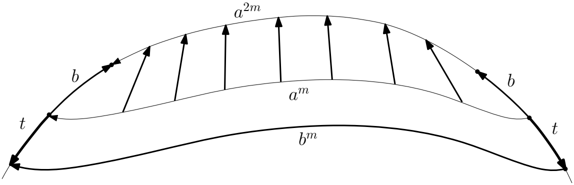

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– While the complexity of the word search problem in a finitely presented group GB is immense, the parameterized complexity shows that this complexity comes purely from the size of the answer, and nothing else.

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Conclusion

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Further developments and current work

– *Word problem in one-relator groups.* It appears that the same approach works to show that the word in arbitrary one-relator groups.

Magnus breakdown: reduce the (extended) word problem in $\langle X \mid r \rangle$ to the extended word problem in $\langle X' \mid r' \rangle$, where $|r'| < |r|$.

Extended word problem in $\langle X \mid r \rangle$ is the membership problem in $\langle Y \rangle$, where $Y \subseteq X$.

– *Word problem in HNN-extensions.* It appears that the same works for a wide class of HNN-extensions. If $G = \langle X \mid R \rangle$, $H, K \leq G$, and $\varphi : H \to K$ is an isomorphism, the corresponding HNN-extension is

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Given group words u, v in X, establish if there is $g \in G$ s.t. $u^g = v$.

What to use as a parameter?
– Length of conjugating element.
– Area of annular diagram. In this case, a result similar to Theorem 1 takes place.

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