On the structure of quasivariety lattices

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Quasivarieties

A *quasivariety* is a class of algebraic structures defined by *quasi-identities*, i.e., universal sentences of the form

$$\forall \overline{x}\ A_0(\overline{x})\ \&\ \ldots\ \&\ A_n(\overline{x}) \longrightarrow A(\overline{x}),$$

where $A_0(\overline{x}), \ldots, A_n(\overline{x}), A(\overline{x})$ are atomic formulas.

For a class $\mathbf{K}$, let $\mathbf{Q}(\mathbf{K})$ denote a least q-variety containing $\mathbf{K}$.

$\mathbf{K}(\sigma)$ denotes the class of all structures of type σ.

Varieties

A *variety* is a class of algebraic structures defined by *identities*, i.e., universal sentences of the form

$$\forall \overline{x}\ A(\overline{x}),$$

where $A(\overline{x})$ is an atomic formula.

For a class $\mathbf{K}$, let $\mathbf{V}(\mathbf{K})$ denote a least variety containing $\mathbf{K}$.

Congruences

For a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ and a structure $\mathcal{A} \in \mathbf{K}(\sigma)$, let

$$\mathrm{Con}_{\mathbf{K}}\,\mathcal{A} = \{\theta \in \mathrm{Con}\,\mathcal{A} \mid \mathcal{A}/\theta \in \mathbf{K}\}.$$

If $\mathbf{K}$ is a quasivariety, then $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}$ is an algebraic lattice for each $\mathcal{A} \in \mathbf{K}(\sigma)$; we call it a *relative congruence lattice*.

If $\mathbf{K}$ is a variety, then $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A} = \mathrm{Con}\,\mathcal{A}$.

Operators

For a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let

- $\mathbf{S}(\mathbf{K})$ denote the class of all structures from $\mathbf{K}(\sigma)$ which are isomorphic to substructures of structures from $\mathbf{K}$.
- $\mathbf{P}(\mathbf{K})$ denote the class of all structures from $\mathbf{K}(\sigma)$ which are isomorphic to Cartesian products of structures from $\mathbf{K}$.
- $\mathbf{P}_s(\mathbf{K})$ denote the class of all structures from $\mathbf{K}(\sigma)$ which are subdirect products of structures from $\mathbf{K}$.
- $\mathbf{H}(\mathbf{K})$ denote the class of all structures from $\mathbf{K}(\sigma)$ which are homomorphic images of structures from $\mathbf{K}$.

[Quasi]variety lattices

For a class $\mathbf{K}$, a subclass $\mathbf{K}' \subseteq \mathbf{K}$ is a *relative subquasivariety of* $\mathbf{K}$, if $\mathbf{K}' = \mathbf{Q}(\mathbf{K}') \cap \mathbf{K}$.

$\mathrm{Lq}(\mathbf{K})$ denotes the lattice of relative subquasivarieties of $\mathbf{K}$.

For a class $\mathbf{K}$, a subclass $\mathbf{K}' \subseteq \mathbf{K}$ is a *relative subvariety of* $\mathbf{K}$, if $\mathbf{K}' = \mathbf{V}(\mathbf{K}') \cap \mathbf{K}$.

$\mathrm{Lv}(\mathbf{K})$ denotes the lattice of relative subvarieties of $\mathbf{K}$.

The Birkhoff–Maltsev problem

Problem (G. Birkhoff (1945); A. I. Maltsev (1966))

Which lattices are isomorphic to lattices of [quasi]varieties?

- The problem is still open and will remain open for quite some while.
- Every finite distributive lattice is a q-lattice (V. I. Tumanov, W. Dziobiak).
- There is a description of algebraic atomistic q-lattices (V. A. Gorbunov *et al.*).
- One of the principal obstacles towards finding a satisfactory general description is the complexity of q-lattices.

Historic remarks

- We consider quasivarieties of finite type.
- Cardinality 2^{ω}: V. P. Belkin, A. I. Budkin, V. A. Gorbunov (1970-ies).
- Elements without covers (and without independent q-bases): V. A. Gorbunov (1977), V. K. Kartashov (1980-ies), S. V. Sizyǐ (1995), A. I. Budkin.
- No nontrivial identities ($\mathrm{FL}(\omega)$ embeds): V. I. Tumanov; V. A. Gorbunov, W. Dziobiak, M. P. Tropin.
- $\mathcal{Q}$-universality: M. V. Sapir (1985); M. E. Adams and W. Dziobiak (1994), V. A. Gorbunov (1995); M. E. Adams and W. Dziobiak, V. Koubek and J. Sichler (after 2000).

Overview of results

- A sufficient condition for existence of 2^ω subclasses with a non-computability property for their q-lattices.
- A sufficient condition for existence of 2^ω subquasivarieties
  - without covers (with no independent q-basis),
  - having an [ω-]independent q-basis,
  - where all finite lattices are representable as Con-lattices and v-lattices.
- A sufficient condition for existence of 2^ω subquasivarieties with
  - the undecidable finite membership problem [the undecidable membership problem for finitely presented structures],
  - the undecidable q-theory.

Overview of results

- A sufficient condition for existence of 2^ω subquasivarieties with a non-computability property for their v-lattices.
- A sufficient condition for existence of 2^ω subquasivarieties with a non-computability property for their Con-lattices.
- Applications.

Q-universality

Definition (M. V. Sapir)

A q-variety $\mathbf{K}$ is *Q-universal* if for any q-variety $\mathbf{M}$, $\mathrm{Lq}(\mathbf{M}) \in \mathbf{HS}(\mathrm{Lq}(\mathbf{K}))$.

M. V. Sapir constructed the first example of a Q-universal q-variety (generated by a single semigroup).

M. E. Adams and W. Dziobiak, as well as V. A. Gorbunov, found a sufficient condition for Q-universality.

AD-classes

Definition (W. Dziobiak(1986); M. E. Adams and W. Dziobiak (1994))

A class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{fin}(\omega)\}$ is an *Adams–Dziobiak class* or an AD-*class*.

(P_0) for each $X \in \mathcal{P}_{fin}(\omega)$, $\mathcal{A}_X$ is I-projective in $\mathbf{Q}(\mathbf{A})$ and the trivial congruence is a dually compact element in $\mathrm{Con}_{\mathbf{Q}(\mathbf{A})} \mathcal{A}_X$;

(P_1) $\mathcal{A}_\varnothing$ is a trivial structure;

(P_2) if $X = Y \cup Z$ then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$;

(P_3) if $X \neq \varnothing$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$ then $X = Y$;

(P_4) if $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$ for some $\mathcal{B}_0$, $\mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$, then there are Y_0, $Y_1 \in \mathcal{P}_{fin}(\omega)$ such that $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0)$, $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1)$, and $X = Y_0 \cup Y_1$,

An AD-class consisting of finite structures is a *finite* AD-*class*.

Theorem (M. E. Adams and W. Dziobiak)

Any q-variety with a finite AD*-class is Q-universal.*

Universality

A q-variety **K** is *universal* if every category of algebraic structures of finite type (or, equivalently, the category **G** of directed graphs with respect to homomorphisms) is isomorphic to a full subcategory of **K**.

If an embedding of **G** into **K** can be realized by a functor which assigns a finite algebraic structure to each finite directed graph, then **K** is *finite-to-finite universal*.

Theorem (M. E. Adams and W. Dziobiak)

Any finite-to-finite universal q-variety contains a finite AD*-class and is therefore Q-universal.*

Theorem (A. M. Nurakunov)

1. *There are continuum many q-varieties* $\mathbf{K}$ *of unars such that set of isomorphism types of finite sublattices of* $\mathrm{Lq}(\mathbf{K})$ *is not computable.*
2. *There are continuum many q-varieties* $\mathbf{K}$ *of Abelian groups with a constant such that set of isomorphism types of finite sublattices of* $\mathrm{Lq}(\mathbf{K})$ *is not computable.*

Theorem (V. I. Tumanov; A. Zamojska-Dzienio and MS)

For a similarity type σ, TFAE:

1. *σ contains either a functional symbol of arity at least 1, or a relation symbol of arity at least 2, or σ is infinite.*
2. $\mathrm{Lq}\big(\mathbf{K}(\sigma)\big)$ *is infinite.*
3. $\mathrm{Lq}\big(\mathbf{K}(\sigma)\big)$ *has cardinality* $2^{|\sigma|+}$.
4. *There are continuum many Q-universal q-varieties* $\mathbf{K}$ *of similarity type σ.*
5. *There are continuum many q-varieties* $\mathbf{K}$ *of similarity type σ such that set of isomorphism types of finite sublattices of* $\mathrm{Lq}(\mathbf{K})$ *is not computable.*

Theorem

Let a q-variety $\mathbf{K}$ *contain an* AD*-class. Then*

1. $\mathbf{K}$ *is Q-universal;*
2. $\mathbf{K}$ *contains continuum many classes* $\mathbf{K}' \subseteq \mathbf{K}$ *such that the set of isomorphism types of finite sublattices of* $\mathrm{Lq}(\mathbf{K}')$ *is not computably enumerable.*

B-classes

Definition (A. V. Kravchenko, A. M. Nurakunov, MS)

Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a q-variety. A class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ is a B-*class with respect to* $\mathbf{M}$, if

(B$_0$) for each nonempty $F \in \mathcal{P}_{fin}(\omega)$, $\mathcal{A}_F$ is finitely presented in $\mathbf{M}$; $\mathcal{A}_\varnothing$ is a trivial structure;

(B$_1$) if $F = G \cup H$ then $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathcal{A}_H)$;

(B$_2$) if $F \neq \varnothing$ and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G)$ then $F = G$;

(B$_3$) if $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_{\{i\}})$ then either $f(\mathcal{A}_F) \cong \mathcal{A}_\varnothing$ or $i \in F$;

(B$_4$) $\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M} \subseteq \mathbf{A}$.

A B-class consisting of finite structures is a *finite* B-*class with respect to* $\mathbf{M}$.

Proposition (A. V. Kravchenko, A. M. Nurakunov, MS)

If **A** is a [finite] B-class with respect to **M**, then **A** is a [finite] AD-class. In particular, if a q-variety contains a B-class, then it is Q-universal and contains. . .

Proposition (A. V. Kravchenko, A. M. Nurakunov, MS)

If a q-variety contains an AD-class or a B-class, then it has 2^ω subquasivarieties containing an AD-class or a B-class.

Independent bases

Let $\mathbf{K} \subseteq \mathbf{M} \subseteq \mathbf{K}(\sigma)$.

Definition

Φ is an *ω-independent q-basis of* $\mathbf{K}$ *relative to* $\mathbf{M}$, if there is a partition $\Phi = \bigcup_{n<\omega} \Phi_n$ such that

$$\mathbf{K} = \mathrm{Mod}(\Phi) \cap \mathbf{M},$$
$$\mathbf{K} \neq \mathrm{Mod}\big(\Phi \setminus \Phi_n\big) \cap \mathbf{M}, \ \ n < \omega.$$

Definition

Φ is an *independent q-basis of* $\mathbf{K}$ *relative to* $\mathbf{M}$, if

$$\mathbf{K} = \mathrm{Mod}(\Phi) \cap \mathbf{M},$$
$$\mathbf{K} \neq \mathrm{Mod}\big(\Phi \setminus \{\varphi\}\big) \cap \mathbf{M}, \ \ \varphi \in \Phi.$$

Proposition (V. A. Gorbunov)

Let $\mathbf{K}_0 \subseteq \mathbf{K}_1 \subseteq \mathbf{K}$ be q-varieties such that

- $\mathbf{K}_0$ has an independent q-basis relative to $\mathbf{K}$.
- $\mathbf{K}_1$ is finitely axiomatizable relative to $\mathbf{K}$.

Then $\mathbf{K}_0$ has an upper cover in $\mathrm{Lq}(\mathbf{K}_1)$.

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let a q-variety $\mathbf{M}$ *contain a* B*-class with respect to* $\mathbf{M}$*. There are continuum many disjoint sets* $\{\mathbf{K}_i \in \mathrm{Lq}(\mathbf{M}) \mid i < 2^\omega\}$ *such that*

- $\mathbf{K}_i$ *does not have an independent q-basis relative to* $\mathbf{M}$ *for each* $i < 2^\omega$*;*
- $\mathbf{K}_i$ *has an* ω*-independent q-basis relative to* $\mathbf{M}$ *for each* $i < 2^\omega$*;*
- $\bigcap_{i<2^\omega} \mathbf{K}_i$ *has an independent q-basis relative to* $\mathbf{M}$*.*

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let a q-variety $\mathbf{M}$ *contain a* B*-class with respect to* $\mathbf{M}$*. Then there are continuum many* $\mathbf{K} \in \mathrm{Lq}(\mathbf{M})$ *which do have an infinite independent q-basis relative to* $\mathbf{M}$*.*

Corollaries

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Each of the following classes **K** *contains a* B*-class with respect to* **K***:*

- *the variety* **U** *of all unars;*
- *the variety* **R** *of all commutative rings with unit;*
- *the q-variety* **G** *of all directed graphs;*
- *the variety* **MV** *of MV-algebras (or Wajsberg algebras);*
- *the variety* $\mathbf{C}_{mn}$ *of Cantor algebras, where* $0 < m < n < \omega$*;*
- *the variety* $\mathbf{A}_c$ *of pointed Abelian groups;*
- *the variety* $\mathbf{M}_{0,1}$ *of modular* $(0, 1)$*-lattices;*

Theorem (continued)

- *a certain subquasivariety of the variety* $\mathbf{U}_0^2$ *of similarity type* $\sigma = \{f, g, c\}$, *where* f, g *are unary functional symbols and* c *is a constant symbol;*
- *the q-variety* $\mathbf{Q}(\mathcal{Z}_3)$, *where* $\mathcal{Z}_3 = \langle \{0, 1, 2\}; \vee, \bar{\ } \rangle$, *where* $\mathcal{Z}_3 = \langle \{0, 1, 2\}; \vee \rangle$ *is a semilattice such that* $0 < 1 < 2$ *and* $\bar{0} = 1$, $\bar{1} = 0$, $\bar{2} = 2$.

A. O. Basheyeva, A. V. Yakovlev

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Each finite-to-finite universal q-variety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ contains a subquasivariety $\mathbf{M} \subseteq \mathbf{K}$ with a finite B-class with respect to $\mathbf{M}$.

Theorem

Each almost finite-to-finite universal q-variety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ *contains continuum many q-varieties which have no upper cover and thus have no independent q-basis relative to* $\mathbf{M}$*, but which have an* ω*-independent q-basis relative to* $\mathbf{M}$*.*

Corollary (A. V. Kravchenko, A. M. Nurakunov, MS)

Each of the following classes $\mathbf{K}$ *contains a* B*-class with respect to some quasivariety* $\mathbf{M} \subseteq \mathbf{K}$*:*

1. *a variety of* $(0,1)$*-lattices that contains a finite non-distributive simple* $(0,1)$*-lattice;*
2. *a variety of modular* $(0,1)$*-lattices that contains* M_3*;*
3. *a q-variety of undirected antireflexive graphs that contains a non-bipartite graph;*
4. *a q-variety of undirected graphs that contains the two-element graph* $\mathcal{A}$ *with the vertex set* $\{a, b\}$ *and the edge set* $\{(a, b), (b, a), (b, b)\}$.

Corollary (A. V. Kravchenko and A. V. Yakovlev; A. V. Kravchenko, A. M. Nurakunov, MS)

TFAE for a q-variety $\mathbf{Q}$ *of undirected antireflexive graphs.*

1. $\mathbf{Q}$ *contains a non-bipartite graph.*
2. $\mathrm{Lq}(\mathbf{Q})$ *has cardinality* 2^ω.
3. $\mathbf{Q}$ *is Q-universal.*
4. $\mathbf{Q}$ *is universal.*
5. $\mathbf{Q}$ *is finite-to-finite universal.*
6. *There are* 2^ω *subclasses* $\mathbf{K} \subseteq \mathbf{Q}$ *such that the set of finite sublattices of* $\mathrm{Lq}(\mathbf{K})$ *is not computable.*
7. *There is* $\mathbf{K} \in \mathrm{Lq}(\mathbf{Q})$ *such that* $\mathrm{Lq}(\mathbf{K})$ *contains* 2^ω *elements which have no upper cover in* $\mathrm{Lq}(\mathbf{K})$.
8. *There is* $\mathbf{K} \in \mathrm{Lq}(\mathbf{Q})$ *such that* $\mathrm{Lq}(\mathbf{K})$ *contains* 2^ω *elements which have no independent q-basis relative to* $\mathbf{K}$ *but which*

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let **K** *be one of the following classes:*

1. *the variety* **Dm** *of differential groupoids;*
2. *the q-variety* $\mathbf{V}_{1,1}$ *of unary algebras whose similarity type consists of two unary functional symbols.*

Then there is no B*-class in* **K***.*
Nonetheless, there are 2^ω *subquasivarieties in* **K** *which have no upper cover in* $\mathrm{Lq}(\mathbf{K})$ *and thus no independent q-basis but which have an* ω*-independent q-basis relative to* **K***. The intersection of those q-varieties does have an independent q-basis.*

C-classes

Definition (A. V. Kravchenko, A. M. Nurakunov, MS)

Let $\mathbf{M}$ be a q-variety. A class $\mathbf{C} = \{\mathcal{C}_n \mid n \in \omega\} \subseteq \mathbf{M}$ of nontrivial structures is a C-*class with respect to* $\mathbf{M}$ if

(C$_1$) $\mathcal{C}_m$ embeds into $\mathcal{C}_n$ if and only if $m = n$;

(C$_2$) $\mathcal{C}_n$ is subdirectly irreducible and finitely presented in $\mathbf{M}$.

A C-class consisting of finite structures is a *finite* C-*class with respect to* $\mathbf{M}$.

Proposition (A. V. Kravchenko, A. M. Nurakunov, MS)

Under certain natural assumptions an AD-class $\mathbf{A}$ contains a C-class with respect to $\mathbf{Q}(\mathbf{A})$.

Effective C-classes

A C-class is *effective* if it satisfies the following conditions:

- there exists a computable sequence $T = \{A_n(x_0, \ldots, x_{l_n}) \mid n < \omega\}$ of monolith-generating atomic formulas for structures in **C**;
- there exists a computable sequence $\Delta = \{(X_n, \Delta_n) \mid n < \omega\}$ of defining pairs for structures in **C**.

Decision problems for **K**

The *finite membership problem*, asks, given a finite structure $\mathcal{A} \in \mathbf{K}(\sigma)$, if it is decidable that $\mathcal{A} \in \mathbf{K}$. In other words, if it is true that the class of finite members of **K** is decidable.

Let $\mathbf{K} \subseteq \mathbf{K}' \subseteq \mathbf{K}(\sigma)$, where $\mathbf{K}'$ is a prevariety. The *membership problem for finitely presented in* $\mathbf{K}'$ *structures* asks, given [the atomic diagram of] a finitely presented in $\mathbf{K}'$ structure $\mathcal{A}$, if it is decidable that $\mathcal{A} \in \mathbf{K}$.

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *be a q-variety, and let* $\mathbf{C} = \{\mathcal{C}_n \mid n < \omega\} \subseteq \mathbf{M}$ *be an effective* C*-class with respect to* $\mathbf{M}$*. Then there are continuum many* $\mathbf{K} \in \mathrm{Lq}(\mathbf{M})$ *such that*

- *the membership problem for finitely presented in* $\mathbf{M}$ *structures is undecidable in* $\mathbf{K}$*;*
- *the q-theory of* $\mathbf{K}$ *is undecidable;*
- $\mathbf{K}$ *has an independent q-basis relative to* $\mathbf{M}$*.*

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a q-variety, and let $\mathbf{C} = \{\mathcal{C}_n \mid n < \omega\} \subseteq \mathbf{M}$ be a finite effective C-class with respect to $\mathbf{M}$. Then there are continuum many $\mathbf{K} \in \mathrm{Lq}(\mathbf{M})$ such that

- *the finite membership problem is undecidable in $\mathbf{K}$;*
- *the q-theory of $\mathbf{K}$ is undecidable;*
- *$\mathbf{K}$ has an independent q-basis relative to $\mathbf{M}$.*

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Each of the following classes **K** *contains a* C*-class with respect to* **K***:*

1. *the variety* **U** *of all unars;*
2. *the variety* **R** *of all commutative rings with unit;*
3. *the q-variety* **G** *of all directed graphs;*
4. *the variety* **MV** *of MV-algebras (or Wajsberg algebras);*
5. *the variety* $\mathbf{C}_{mn}$ *of Cantor algebras, where* $0 < m < n < \omega$*;*
6. *the variety* $\mathbf{A}_c$ *of pointed Abelian groups;*
7. *a variety of* $(0, 1)$*-lattices that contains a finite non-distributive simple* $(0, 1)$*-lattice;*
8. *a variety of modular* $(0, 1)$*-lattices that contains* M_3*;*

Theorem (continued)

9 *a q-variety of undirected antireflexive graphs that contains a non-bipartite graph;*

10 *a q-variety of undirected graphs that contains the two-element graph $\mathcal{A}$ with the vertex set $\{a, b\}$ and the edge set $\{(a, b), (b, a), (b, b)\}$.*

11 *the variety* **Dm** *of differential groupoids;*

12 *a q-variety* $\mathbf{V}_{1,1}$ *of unary algebras whose similarity type consists of two unary functional symbols;*

13 *the q-variety* $\mathbf{Q}(\mathcal{Z}_3)$, *where* $\mathcal{Z}_3 = \langle\{0, 1, 2\}; \vee, ^{-}\rangle$, *where* $\mathcal{Z}_3 = \langle\{0, 1, 2\}; \vee\rangle$ *is a semilattice such that* $0 < 1 < 2$ *and* $\bar{0} = 1,\ \bar{1} = 0,\ \bar{2} = 2.$

Relative congruence lattices

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let $\mathbf{M}$ *be a q-variety and let* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ *be a* B*-class with respect to* $\mathbf{M}$*. Then for any finite lattice* L*, there is a set* $F \in \mathcal{P}_{fin}(\omega)$ *and a q-variety* $\mathbf{K} \subseteq \mathbf{Q}(\mathbf{A})$ *such that* $\mathcal{A}_F \in \mathbf{K}$ *and* $\mathrm{Con}_{\mathbf{K}}\, \mathcal{A}_F \cong L$*.*

This theorem applies to many classes listed above.

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let $\mathbf{M}$ *be one of the following classes:*

1. *the variety* $\mathbf{U}$ *of all unars;*
2. *the variety* $\mathbf{R}$ *of all commutative rings with unit;*
3. *the quasivariety* $\mathbf{D}$ *of all directed graphs;*
4. *the variety* $\mathbf{A}_c$ *of pointed Abelian groups;*
5. *a certain subquasivariety of the variety* $\mathbf{U}_0^2$ *of similarity type* $\sigma = \{f, g, c\}$, *where* f, g *are unary functional symbols and* c *is a constant symbol.*

Then there are 2^ω *subquasivarieties* $\mathbf{K} \subseteq \mathbf{M}$ *and* $\mathcal{A} \in \mathbf{K}$ *such that the finite membership problem is undecidable for both* $\mathbf{K}$ *and* $\mathbf{S}(\mathrm{Con}_{\mathbf{K}}\,\mathcal{A})$.

Theorem (A. V. Kravchenko and MS)

Let $\mathbf{M}$ *be one of the following classes:*

1. *the variety* $\mathbf{Dm}$ *of differential groupoids;*
2. *an axiomatizable class* $\mathbf{W} \subseteq \mathbf{V}_{1,1}$ *of unary algebras whose similarity type consists of two unary functional symbols.*

For every finite lattice L *where the greatest element is* $\vee$*-irreducible, there is a* $\mathbf{M}$*-quasivariety* $\mathbf{K}$ *and* $\mathcal{A} \in \mathbf{K}$ *such that* $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A} \cong L$*.*

Theorem (A. V. Kravchenko and MS)

The following statements hold.

1. *There are* 2^ω *quasivarieties* $\mathbf{K} \subseteq \mathbf{Dm}$ *such that the finite membership problem is undecidable for both* $\mathbf{K}$ *and* $\mathbf{S}(\mathrm{Con}_{\mathbf{K}}\,\mathcal{A})$, *where* $\mathcal{A} \in \mathbf{K}$, *and* $\mathrm{Th}_q(\mathbf{K})$ *is also undecidable.*
2. *There are* 2^ω *axiomatizable classes* $\mathbf{K} \subseteq \mathbf{W}$ *such that the finite membership problem is undecidable for both* $\mathbf{K}$ *and* $\mathbf{S}(\mathrm{Con}_{\mathbf{K}}\,\mathcal{A})$, *where* $\mathcal{A} \in \mathbf{K}$, *and* $\mathrm{Th}_q(\mathbf{K})$ *is also undecidable.*

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS)

Let **M** *be one of the following classes:*

1. *the variety* **R** *of all commutative rings with unit;*
2. *the variety* **U** *of all unars;*
3. *the variety* $\mathbf{A}_c$ *of pointed Abelian groups;*
4. *a certain subquasivariety of the variety* $\mathbf{U}_0^2$ *of similarity type* $\sigma = \{f, g, c\}$, *where* f, g *are unary functional symbols and* c *is a constant symbol;*
5. *the variety* **MV** *of all MV-algebras (or Wajsberg algebras);*
6. *the variety* $\mathbf{M}_{0,1}$ *of modular* $(0, 1)$*-lattices.*

Then there are 2^ω *classes* $\mathbf{K} \subseteq \mathbf{M}$ *such that the finite membership problem is undecidable for both* **K** *and* $\mathbf{S}(\mathrm{Lv}(\mathbf{K}))$.
In (2)–(4), the conclusion holds for some quasivariety **K**.

Theorem (A. V. Kravchenko and MS)

For every finite lattice L where the least element is $\wedge$-irreducible, there is a quasivariety $\mathbf{K} \subseteq \mathbf{Dm}$ such that $\mathrm{Lv}(\mathbf{K}) \cong L$.

Theorem (A. V. Kravchenko and MS)

There are 2^ω prevarieties $\mathbf{K} \subseteq \mathbf{Dm}$ such that the finite membership problem is undecidable for both $\mathbf{K}$ and $\mathbf{S}(\mathrm{Lv}(\mathbf{K}))$, and $\mathrm{Th}_q(\mathbf{K})$ is undecidable.

An old problem

Is the variety of groups Q-universal?

Problem

Does the variety of groups contain a B-class with respect to some subquasivariety?

Theorem (A. I. Budkin, 2019)

Let $\mathbf{G}_0$ denote the quasivariety of all torsion-free groups. There is a set $\{\mathbf{K}_i \in \mathrm{Lq}(\mathbf{G}_0) \mid i < 2^\omega\}$ such that

- $\mathbf{K}_i$ *does not have an independent q-basis relative to* $\mathbf{G}_0$ *for each* $i < 2^\omega$;
- $\mathbf{K}_i$ *has an* ω*-independent q-basis relative to* $\mathbf{G}_0$ *for each* $i < 2^\omega$;
- $\bigcap_{i<2^\omega} \mathbf{K}_i$ *has an independent q-basis relative to* $\mathbf{G}_0$.

Theorem (A. I. Budkin, 2019)

Let $\mathbf{N}_2$ denote the variety of all 2-nilpotent groups. There is a set $\{\mathbf{K}_i \in \mathrm{Lq}(\mathbf{N}_2) \mid i < 2^\omega\}$ such that

- $\mathbf{K}_i$ *does not have an independent q-basis relative to* $\mathbf{N}_2$ *for each* $i < 2^\omega$*;*
- $\mathbf{K}_i$ *has an* ω*-independent q-basis relative to* $\mathbf{N}_2$ *for each* $i < 2^\omega$*;*
- $\bigcap_{i<2^\omega} \mathbf{K}_i$ *has an independent q-basis relative to* $\mathbf{N}_2$*.*

Problem

Does there exist a weaker sufficient condition which still guarantees the high complexity of q-lattices?